On the universal embedding of the near hexagon related to the extended ternary Golay code

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Abstract

Let $E_1$ be the near hexagon on 729 points related to the extended ternary Golay code. We prove in an entirely geometric way that the generating and embedding ranks of $E_1$ are equal to 24. We also study the structure of the universal embedding $\tilde{e}$ of $E_1$. More precisely, we consider several nice subgeometries $A$ of $E_1$ and determine which kind of embedding $\tilde{e}_A$ is, where $\tilde{e}_A$ is the embedding of $A$ induced by $\tilde{e}$.

Keywords: generating rank, embedding rank, universal embedding, near hexagon, extended ternary Golay code
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1 Introduction

In the literature, the embedding and generating ranks have been determined for several classes of point-line geometries, see Cooperstein [7] for a survey of the most important results obtained on this topic before the year 2003. In the present paper, we determine in an entirely geometric way the embedding and generating ranks of the near hexagon $E_1$ on 729 points which is related to the extended ternary Golay code. Previously, only the embedding rank of that geometry was known. This embedding rank was determined in Brouwer et al. [4, p. 350] with the aid of a computer and in Yoshiara [16] with some group theoretical argument involving the Leech lattice.

Theorem 1.1 The embedding and generating ranks of $E_1$ are equal to 24.

The near hexagon $E_1$ has many subgeometries which are isomorphic to the point-line geometry $A^*$ which we are going to define now. Embed $PG(4,3)$ as a hyperplane in $PG(5,3)$ and let $X$ be a set of 6 points of $PG(4,3)$ no five of which are contained in a hyperplane of $PG(4,3)$. Then $A^*$ is defined as follows:

- the points of $A^*$ are the points of $PG(5,3)$ not contained in $PG(4,3)$;
the lines of $\mathcal{A}^*$ are the lines of $\text{PG}(5, 3)$ not contained in $\text{PG}(4, 3)$ which contain a unique point of $X$;
• incidence is the one derived from $\text{PG}(5, 3)$.

Any subgeometry of $\mathcal{E}_1$ which is isomorphic to $\mathcal{A}^*$ is called a special subgeometry of $\mathcal{E}_1$.

For every $i \in \{1, 2, \ldots, 6\}$, the near hexagon $\mathcal{E}_1$ also has subgeometries which are isomorphic to the Hamming near $2i$-gon $H(i, 3)$. In Section 3.1, we will determine all special subgeometries of $\mathcal{E}_1$ as well as all subgeometries isomorphic to $H(i, 3)$, $i \in \{1, 2, \ldots, 6\}$.

An essential step in the proof that $\mathcal{E}_1$ can be generated by 24 points (see Theorem 1.1) is the proof that any special subgeometry of $\mathcal{E}_1$ can be generated by 22 points. This latter fact will also allow us to determine the embedding and generating ranks of $\mathcal{A}^*$.

**Theorem 1.2** The embedding and generating ranks of $\mathcal{A}^*$ are equal to 22.

We are also interested in the structure of the universal embedding $\tilde{e}$ of $\mathcal{E}_1$. More precisely, we are interested in the following kind of problem.

Suppose $\mathcal{A}$ is a subgeometry of $\mathcal{E}_1$ and $\tilde{e}_\mathcal{A}$ is the embedding of $\mathcal{A}$ induced by $\tilde{e}$. What kind of embedding is $\tilde{e}_\mathcal{A}$?

We will give an answer to the above question in the case that $\mathcal{A}$ is a special subgeometry or a subgeometry isomorphic to $H(i, 3)$ for some $i \in \{1, 2, 3, 4\}$.

**Theorem 1.3** Let $\tilde{e}$ denote the universal embedding of $\mathcal{E}_1$, let $\mathcal{A}$ be a subgeometry of $\mathcal{E}_1$ which is either a special subgeometry or a subgeometry isomorphic to $H(i, 3)$ for some $i \in \{1, 2, 3, 4\}$. Let $X$ denote the point set of $\mathcal{A}$. Then the following holds.

1. The projective embedding of $\mathcal{A}$ induced by $\tilde{e}$ is isomorphic to the universal embedding of $\mathcal{A}$.
2. A point $x$ of $\mathcal{E}_1$ belongs to $X$ if and only if $\tilde{e}(x) \in \langle \tilde{e}(X) \rangle$.

A description of the universal embedding of $\mathcal{E}_1$ was given in Yoshiara [16, Section 3.2]. This universal embedding was realized in the Leech lattice modulo 2. In Section 3.2, we give an explicit description of the universal embeddings of the Hamming near polygons $H(n, 3)$, $n \geq 1$.

## 2 Basic notions

### 2.1 Near polygons

A near polygon is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$, $I \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point $x \in \mathcal{P}$ and every line $L \in \mathcal{L}$ there exists a unique point on $L$ nearest to $x$. Here, distances are measured in the collinearity graph $\Gamma$ of $\mathcal{S}$. If $d \in \mathbb{N}$ is the diameter of $\Gamma$, then the near polygon is called a near $2d$-gon. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles.
A finite near $2d$-gon $S$ with $d \geq 2$ is called regular if its collinearity graph is a so-called distance-regular graph (Brouwer et al. [5]). This implies that there exist constants $s, t, t_i$ ($i \in \{2, \ldots, d - 1\}$) such that every line is incident with precisely $s + 1$ points, every point is incident with precisely $t + 1$ lines and for every two points $x$ and $y$ at distance $i \in \{2, \ldots, d - 1\}$ from each other, there are precisely $t_i + 1$ lines through $y$ containing a (necessarily unique) point at distance $i - 1$ from $x$. We call $(s, t, t_2, t_3, \ldots, t_{d-1})$ the parameters of $S$.

Let $n, k \in \mathbb{N} \setminus \{0\}$ with $k \geq 2$ and put $A := \{1, 2, \ldots, k\}$. Let $H(n, k)$ denote the point-line geometry whose points are the elements of the cartesian power $A^n$ and whose lines are all the sets of the form $\{a_1\} \times \cdots \times \{a_{i-1}\} \times A \times \{a_{i+1}\} \times \cdots \times \{a_n\}$, where $i$ is some element of $\{1, 2, \ldots, n\}$ and $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ are some elements of $A$ (natural incidence). The point-line geometry $H(n, k)$ is a near $2n$-gon. It is called a Hamming near polygon.

Let $\mathbb{F}_3^{12}$ denote the 12-dimensional vector space over the field $\mathbb{F}_3$ of order 3 whose vectors are row matrices of length 12 with entries in $\mathbb{F}_3$. The 6 rows of the matrix

$$M := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 1 & -1
\end{bmatrix}$$

generate a 6-dimensional subspace $G_{12}$ of $\mathbb{F}_3^{12}$ which is called the extended ternary Golay code. By deleting one coordinate position, one gets a code (a subspace of $\mathbb{F}_3^{|E_i|}$) which was discovered by Golay [12]. Let $E_1$ be the point-line geometry whose points are all the cosets of $G_{12}$ and whose lines are all the triples of the form $\{v + G_{12}, \bar{v} + \bar{e}_i + G_{12}, \bar{v} - \bar{e}_i + G_{12}\}$, with incidence being containment. Here, $\bar{v}$ is some vector of $\mathbb{F}_3^{12}$ and $\bar{e}_i, i \in \{1, 2, \ldots, 12\}$, denotes the row matrix all of whose entries are 0 except for the $i$-th one which is equal to 1. Shult and Yanushka [15, pp. 30–33] proved that $E_1$ is a regular near hexagon with parameters $(s, t, t_2) = (2, 11, 1)$. Brouwer [3] proved that $E_1$ is the unique regular near hexagon with parameters $(s, t, t_2) = (2, 11, 1)$. Every two points of $E_1$ at distance 2 from each other are contained in a unique $(3 \times 3)$-subgrid, called a quad, see Shult and Yanushka [15, Proposition 2.5]. If $L_1$ and $L_2$ are two lines meeting in a unique point, then also $L_1$ and $L_2$ are contained in a unique quad.

Another model for the near hexagon $E_1$ was described in De Bruyn and De Clerck [11]. Let $\Pi_\infty$ be a hyperplane of the projective space PG$(6, 3)$. For every set $K$ of points of $\Pi_\infty$, let $T_5^*(K)$ denote the point-line geometry whose points are the points of PG$(6, 3) \setminus \Pi_\infty$ and whose lines are those lines of PG$(6, 3)$ not contained in $\Pi_\infty$ which intersect $\Pi_\infty$ in a point of $K$ (natural incidence). After fixing some reference system in $\Pi_\infty$, the 12 columns of the matrix $M$ define a set $K^*$ of 12 points of $\Pi_\infty$. By De Bruyn [9, Theorem 6.62(b)], the point-line geometry $T_5^*(K^*)$ is a regular near hexagon with parameters $(s, t, t_2) = (2, 11, 1)$. Hence, $T_5^*(K^*) \cong E_1$. 

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The set $K^*$ satisfies several nice properties, see e.g. Coxeter [8]. Among them, we have the following ones which are of interest for the present paper.

- If $Y$ is a set of $i \in \{1, 2, 3, 4\}$ points of $K^*$, then $\dim(\langle Y \rangle) = i - 1$ and $\langle Y \rangle \cap K^* = Y$.
- If $Y$ is a set of 5 points of $K^*$, then $\langle Y \rangle$ is a hyperplane of $\Pi_\infty$ and $\langle Y \rangle$ contains precisely 6 points of $K^*$. Hence, if $B$ denotes the set of all sets $B$ of size 6 which can be obtained by intersecting $K^*$ with a hyperplane of $\Pi_\infty$, then the point-line geometry with point set $K^*$, line set $B$ and natural incidence relation is a Steiner system $S(5, 6, 12)$. There is only one Steiner system with these parameters, see Beth, Jungnickel and Lenz [1, p. 240, Corollary 2.6]. In fact, this Steiner system is one of the small Witt designs.

- It is known (see [1, p. 238]) that the complement of a block of $S(5, 6, 12)$ is again a block of $S(5, 6, 12)$. Hence, if $K^* = \{x_1, x_2, \ldots, x_{12}\}$ such that $\langle x_1, x_2, \ldots, x_6 \rangle$ is a hyperplane of $\Pi_\infty$, then also $\langle x_7, x_8, \ldots, x_{12} \rangle$ is a hyperplane of $\Pi_\infty$.

2.2 Generating rank, embedding rank and universal embedding

Let $S = (\mathcal{P}, \mathcal{L}, 1)$ be a partial linear space. A subspace of $S$ is a set of points which contains all the points of a line as soon as it contains at least two points of it. A hyperplane of $S$ is a subspace distinct from $\mathcal{P}$ which meets each line. If $X$ is a subspace of $S$, then $\tilde{X}$ denotes the subgeometry of $S$ defined on the point set $X$ by those lines of $S$ which have all their points in $X$. If $X$ is a set of points of $S$, then $[X]$ denotes the smallest subspace of $S$ containing the set $X$. We call $[X]$ the subspace of $S$ generated by $X$. If $[X] = \mathcal{P}$, then we will also say that $X$ generates $S$ or that $X$ is a generating set of $S$. The minimal size of a generating set of $S$ is called the generating rank of $S$ and is denoted by $gr(S)$.

An embedding $e$ of $S$ into a projective space $\Sigma$ is an injective mapping $e$ from $\mathcal{P}$ to the point set of $\Sigma$ satisfying: (i) $\langle e(\mathcal{P}) \rangle_\Sigma = \Sigma$; (ii) $e(L) := \{e(x) \mid x \in L\}$ is contained in a line of $\Sigma$ for every line $L$ of $S$. The embedding $e$ is called full if $e(L)$ is a line of $\Sigma$ for every line $L$ of $S$. If $n$ is the maximal dimension of a projective space in which $S$ has a full embedding, then the number $er(S) := n + 1$ is called the embedding rank of $S$. Certainly, $er(S)$ is only defined when $S$ admits a full embedding, in which case it holds that $er(S) \leq gr(S)$. If $e$ is a full embedding of $S$ into a projective space $\Sigma$ and if $\Pi$ is a hyperplane of $\Sigma$, then $e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a hyperplane of $S$. Any hyperplane of $S$ which can be obtained in this way is said to arise from $e$.

Suppose $S = (\mathcal{P}, \mathcal{L}, 1)$ is a fully embeddable point-line geometry with three points on each line. Then by Ronan [13], $S$ admits the so-called universal embedding and every hyperplane of $S$ arises from this embedding. We now give a description of this universal embedding. Let $V$ be a vector space over the field $\mathbb{F}_2$ of order 2 with a basis $B$ whose vectors are indexed by the elements of $\mathcal{P}$, say $B = \{\overline{v}_x \mid x \in \mathcal{P}\}$. Let $W$ denote the subspace of $V$ generated by all vectors $\overline{v}_{x_1} + \overline{v}_{x_2} + \overline{v}_{x_3}$ where $\{x_1, x_2, x_3\}$ is a line of $S$. Then the map $x \in \mathcal{P} \mapsto \{\overline{v}_x + W, W\}$ defines a full embedding of $S$ into the projective space $\text{PG}(V/W)$ which is isomorphic to the universal embedding of $S$. We have $er(S) = \dim(V/W) = \dim(V) - \dim(W)$.

It is known that every dense near polygon with three points per line admits a full projective embedding (Brouwer and Shpectorov [6, Proposition 3.1(ii)]; De Bruyn [10,
Proposition 3.11]). In particular, this holds for the near hexagon $\mathbb{E}_1$.

3 Some useful results

The aim of this section is to study some subgeometries of $\mathbb{E}_1$, to determine the generating rank, embedding rank and universal embedding of every Hamming near polygon $H(n, 3)$, $n \geq 2$, and to derive a lower bound for $er(\mathbb{E}_1)$.

3.1 Subgeometries of $\mathbb{E}_1$

As in Section 2.1, let $\Pi_\infty$ be a hyperplane of $\text{PG}(6, 3)$ and let $\mathcal{K}^*$ be a set of 12 points of $\Pi_\infty$ defined by the columns of the matrix $M$. Then $T_5^*(\mathcal{K}^*) \cong \mathbb{E}_1$. If $L$ is a line of $\text{PG}(6, 3)$ not contained in $\Pi_\infty$, then the unique point of $L \cap \Pi_\infty$ is called the point at infinity of $L$. Let $V$ be a 7-dimensional vector space over $\mathbb{F}_3$ such that $\text{PG}(6, 3) = \text{PG}(V)$.

Let $Y$ be a set of $i \in \{1, 2, \ldots, 6\}$ linearly independent points of $\mathcal{K}^*$ and let $\alpha$ be an $i$-dimensional subspace of $\text{PG}(6, 3)$ which intersects $\Pi_\infty$ in the subspace $(Y)$. Then $S_\alpha := \alpha \setminus \Pi_\infty$ is a subspace of the near hexagon $T_5^*(\mathcal{K}^*)$. Let $\mathcal{A}$ be the subgeometry of $T_3^*(\mathcal{K}^*)$ defined on the set $S_\alpha$ by those lines of $T_5^*(\mathcal{K}^*)$ whose point at infinity belongs to $Y$. Notice that if $i \in \{1, 2, 3, 4\}$, then $\mathcal{A} = \overline{S}_\alpha$. By coordinatizing with respect to a basis $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_7\}$ of $V$ for which $Y = \{\langle \bar{e}_1 \rangle, \langle \bar{e}_2 \rangle, \ldots, \langle \bar{e}_i \rangle\}$ and $\langle \bar{e}_{i+1} \rangle \in \alpha \setminus \Pi_\infty$, we readily see that the geometry $\mathcal{A}$ must be isomorphic to $H(i, 3)$.

Theorem 3.1 Every subgeometry $\mathcal{A}$ of $T_5^*(\mathcal{K}^*)$ isomorphic to $H(i, 3)$, $i \in \{1, 2, \ldots, 6\}$, is obtained in the above-described way.

Proof. Let $x$ be an arbitrary point of $\mathcal{A}$, let $L_1, L_2, \ldots, L_i$ denote the $i$ lines of $\mathcal{A}$ through $x$ and let $y_j, j \in \{1, 2, \ldots, i\}$, denote the point at infinity of $L_j$. Put $\alpha := \langle y_1, y_2, \ldots, y_i, x \rangle$. If $i \leq 5$, then the points $y_1, y_2, \ldots, y_i$ are linearly independent and hence $\alpha$ has dimension $i$. If $i = 6$, then $\alpha$ has dimension 5 or 6. The lines of $\mathcal{A}$ can be partitioned in a natural way into parallel classes$^1$. Let $C_j, j \in \{1, 2, \ldots, i\}$, denote the unique parallel class which contains the line $L_j$. If $L$ and $L'$ are two lines of $\mathcal{A}$, then we write $L \sim L'$ if $L$ and $L'$ are two disjoint lines of $\mathcal{A}$ which are contained in a $(3 \times 3)$-subgrid.

Claim. Every line of $C_j$ has $y_j$ as point at infinity.

Proof. Since $y_j$ is the point at infinity of the line $L_j \in C_j$, it suffices to prove that any two distinct lines $K_1$ and $K_2$ of $C_j$ have the same point at infinity. Now, for two lines $K_1, K_2 \in C_j$, there exist lines $M_0, M_1, \ldots, M_k \in C_j$ (for some $k \in \mathbb{N}$) such that $K_1 = M_0 \sim M_1 \sim \cdots \sim M_k = K_2$. So, it suffices to consider lines $K_1, K_2 \in C_j$ for which $K_1 \sim K_2$. Let $\mathcal{G}$ denote the unique $(3 \times 3)$-subgrid containing $K_1$ and $K_2$. Let $K_3$ denote a line of $\mathcal{G}$ meeting $K_1$ and $K_2$, let $z_j, j \in \{1, 2, 3\}$, denote the point at infinity of the line $K_j$ and let $u$ denote the unique point in $K_1 \cap K_3$. Put $\beta := \langle z_1, z_3, u \rangle$. Then $S_\beta := \beta \setminus \Pi_\infty$.

$^1$Two lines $L_1$ and $L_2$ of a Hamming near polygon are called parallel if each point of $L_1$ has the same distance to $L_2$. 

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is a subspace of $T_5^*(\mathcal{K}^*)$ and $\tilde{S}_\beta$ is a $(3 \times 3)$-subgrid and hence a quad. Since there is only one quad through $K_1$ and $K_3$, we necessarily have $\mathcal{G} = \tilde{S}_\beta$. Hence $K_2$ is contained in $\tilde{S}_\beta$ and has either $z_1$ or $z_3$ as point at infinity. Since $K_2$ and $K_3$ meet, $K_2$ must have $z_1$ as point at infinity. So, $K_1$ and $K_2$ have the same point at infinity. (qed)

By the previous claim and the connectedness of $\mathcal{A}$, it now follows that every point of $\mathcal{A}$ belongs to $\alpha \setminus \Pi_\infty$. If $i \leq 5$, then since $\mathcal{A}$ and $\alpha \setminus \Pi_\infty$ have the same number of points, namely 3, we see that $\alpha \setminus \Pi_\infty$ equals the point set of $\mathcal{A}$. If $i = 6$, then since $\mathcal{A}$ contains 3 points, the subspace $\alpha$ must have dimension 6. So, also in this case $\alpha \setminus \Pi_\infty$ equals the point set of $\mathcal{A}$. Moreover, the points $y_1, y_2, \ldots, y_6$ are linearly independent. Taking the above Claim into account, we now see that $\mathcal{A}$ can be obtained as described before this theorem.

Let $Z$ be a set of 6 points of $\mathcal{K}^*$ such that $\langle Z \rangle$ is a hyperplane of $\Pi_\infty$. Let $\mathcal{H}$ be a 5-dimensional subspace of PG(6, 3) which intersects $\Pi_\infty$ in the subspace $\langle Z \rangle$. Then $S_\beta := \mathcal{H} \cap \Pi_\infty$ is a subspace of the near hexagon $T_5^*(\mathcal{K}^*)$ and $\tilde{S}_\beta$ is a special subgeometry of $T_5^*(\mathcal{K}^*)$.

**Theorem 3.2** Every special subgeometry of $T_5^*(\mathcal{K}^*)$ is obtained in the above-described way.

**Proof.** Let $\mathcal{A}$ be a special subgeometry of $T_5^*(\mathcal{K}^*)$. Then there exists a subgeometry $\mathcal{A}' \cong H(5, 3)$ of $\mathcal{A}$ whose point set $\mathcal{P}$ equals the point set of $\mathcal{A}$. Now, $\mathcal{A}'$ is also a subgeometry of $T_5^*(\mathcal{K}^*)$. So, by Theorem 3.1 there exists a set $Z$ of 5 points of $\mathcal{K}^*$ and a 5-dimensional subspace $\mathcal{H}$ of PG(6, 3) such that $\langle Z \rangle = \Pi_\infty \cap \mathcal{H}$ and $\mathcal{P} = \mathcal{H} \cap \Pi_\infty$. Since $\langle Z \rangle$ contains precisely 6 points of $\mathcal{K}^*$, we necessarily have $\mathcal{A} \cong \tilde{S}_\beta$. ■

### 3.2 Generating and embedding Hamming near polygons

Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, I_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, I_2)$ be two partial linear spaces. Without loss of generality, we may suppose that the sets $\mathcal{P}_1 \times \mathcal{L}_2$ and $\mathcal{L}_1 \times \mathcal{P}_2$ are disjoint. From $\mathcal{S}_1$ and $\mathcal{S}_2$, a new partial linear space $\mathcal{S}_1 \times \mathcal{S}_2 = (\mathcal{P}, \mathcal{L}, I)$ can be derived which is called the **direct product** of $\mathcal{S}_1$ and $\mathcal{S}_2$. The point set $\mathcal{P}$ of $\mathcal{S}_1 \times \mathcal{S}_2$ is equal to the cartesian product $\mathcal{P}_1 \times \mathcal{P}_2$ and the line set $\mathcal{L}$ of $\mathcal{S}_1 \times \mathcal{S}_2$ is equal to $\mathcal{P}_1 \times \mathcal{L}_2$ or $\mathcal{L}_1 \times \mathcal{P}_2$. A point $(x, y)$ of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(z, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = z$ and $(y, L) \in I_2$. The point $(x, y)$ of $\mathcal{S}_1 \times \mathcal{S}_2$ is incident with the line $(M, u) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $(x, M) \in I_1$ and $y = u$. If $\mathcal{S}_i$, $i \in \{1, 2\}$, is a near $2n_i$-gon, then $\mathcal{S}_1 \times \mathcal{S}_2$ is a near $2(n_1 + n_2)$-gon. If $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are three partial linear spaces, then $\mathcal{S}_1 \times \mathcal{S}_2 \cong \mathcal{S}_2 \times \mathcal{S}_1$ and $(\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{S}_3 \cong \mathcal{S}_1 \times (\mathcal{S}_2 \times \mathcal{S}_3)$. So, the direct product $\mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_k$ of $k \geq 2$ partial linear spaces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ is well-defined. If we denote by $\mathcal{L}_k$ the line of size $k \geq 2$, then the direct product $\mathcal{L}_k \times \mathcal{L}_k \times \cdots \times \mathcal{L}_k$ of $n \geq 2$ isomorphic copies of $\mathcal{L}_k$ is isomorphic to the Hamming near $2n$-gon $H(n, k)$. 

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Lemma 3.3 Let \( S_1 = (\mathcal{P}_1, \mathcal{L}_1, I_1) \) and \( S_2 = (\mathcal{P}_2, \mathcal{L}_2, I_2) \) be two partial linear spaces. If \( X_i, i \in \{1, 2\} \), is a generating set of points of \( S_i \), then \( X_1 \times X_2 \) is a generating set of points of \( S_1 \times S_2 \).

Proof. The partial linear space \( S_1 \times S_2 \) has subgeometries isomorphic to \( S_2 \), namely for every \( u \in \mathcal{P}_1 \), \( \{u\} \times \mathcal{P}_2 \) is a subspace of \( S_1 \times S_2 \) and \( \{u\} \times \mathcal{P}_2 \cong S_2 \). We observe the following:

(a) Since \( X_2 \) is a generating set of points of \( S_2 \), \( \{x_1\} \times X_2 = \{x_1\} \times \mathcal{P}_2 \) for every \( x_1 \in X_1 \). Hence, \( \{x_1\} \times \mathcal{P}_2 \subseteq [X_1 \cup X_2] \) for every \( x_1 \in X_1 \).

(b) Suppose \( u_1, u_2 \) and \( u_3 \) are three mutually distinct points contained on some line of \( S_1 \). Then \( \{u_1\} \times \mathcal{P}_2 \cup \{u_2\} \times \mathcal{P}_2 \subseteq [X_1 \cup X_2] \) implies that \( \{u_3\} \times \mathcal{P}_2 \subseteq [X_1 \cup X_2] \). For, every point \( (u_3, v) \in \{u_3\} \times \mathcal{P}_2 \) is contained on the line joining the point \( (u_1, v) \in \{u_1\} \times \mathcal{P}_2 \) with the point \( (u_2, v) \in \{u_2\} \times \mathcal{P}_2 \).

The lemma follows from (a) and (b) above. \( \blacksquare \)

Corollary 3.4 Let \( n, k \in \mathbb{N} \setminus \{0\} \) with \( k \geq 2 \). Then the Hamming near polygon \( H(n, k) \) has a generating set of size \( 2^n \).

Proof. This follows from Lemma 3.3 taking into account that \( H(n, k) \) is isomorphic to the direct product of \( n \) copies of \( \mathbb{L}_k \) and that \( \mathbb{L}_k \) is generated by two points. \( \blacksquare \)

Let \( S_1 = (\mathcal{P}_1, \mathcal{L}_1, I_1) \) and \( S_2 = (\mathcal{P}_2, \mathcal{L}_2, I_2) \) be two partial linear spaces. Let \( e_i, i \in \{1, 2\} \), be a (not necessarily full) projective embedding of \( S_i \) into \( \text{PG}(V_i) \), where \( V_i \) is some vector space over a field \( \mathbb{K} \). For every point \( p \) of \( S_i \), \( i \in \{1, 2\} \), let \( \theta_i(p) \) denote a vector of \( V_i \) such that \( e_i(p) = \langle \theta_i(p) \rangle \). Then the map \( e_1 \otimes e_2 : \mathcal{P}_1 \times \mathcal{P}_2 \to \text{PG}(V_1 \otimes V_2) : (p_1, p_2) \mapsto \langle \theta_1(p_1) \otimes \theta_2(p_2) \rangle \) is a projective embedding of \( S_1 \times S_2 \) into \( \text{PG}(V_1 \otimes V_2) \). If \( e_1 \) and \( e_2 \) are full, then also \( e_1 \otimes e_2 \) is full.

Proposition 3.5 (1) Let \( n, k \in \mathbb{N} \setminus \{0\} \) with \( k \geq 2 \). Then the Hamming near polygon \( H(n, k) \) has generating rank \( 2^n \).

(2) Let \( V \) be a 2-dimensional vector space over \( \mathbb{F}_2 \) and let \( e \) be a full projective embedding of \( \mathbb{L}_3 \) into \( \text{PG}(V) \). Then \( e \otimes e \otimes \cdots \otimes e \) (\( n \) times) is isomorphic to the universal embedding of \( H(n, 3) \cong \mathbb{L}_3 \times \mathbb{L}_3 \times \cdots \times \mathbb{L}_3 \).

Proof. Let \( \mathbb{K} \) be a field for which \( k \leq |\mathbb{K}| + 1 \), with \( \mathbb{K} = \mathbb{F}_2 \) if \( k = 3 \). Let \( V \) be a 2-dimensional vector space over \( \mathbb{K} \) and let \( e \) be a projective embedding of \( \mathbb{L}_k \) into \( \text{PG}(V) \). Since \( e \otimes e \otimes \cdots \otimes e \) is a projective embedding of \( H(n, k) \cong \mathbb{L}_k \times \mathbb{L}_k \times \cdots \times \mathbb{L}_k \) into \( \text{PG}(V \otimes V \otimes \cdots \otimes V) \), the generating rank of \( H(n, k) \) is at least \( \dim(V \otimes V \otimes \cdots \otimes V) = 2^n \). Corollary 3.4 now implies that the generating rank of \( H(n, k) \) is equal to \( 2^n \). In the case \( k = 3 \) (and \( \mathbb{K} = \mathbb{F}_2 \)), we can say more, namely, that \( e \otimes e \otimes \cdots \otimes e \) must be isomorphic to the universal embedding of \( H(n, 3) \cong \mathbb{L}_3 \times \mathbb{L}_3 \times \cdots \times \mathbb{L}_3 \). \( \blacksquare \)
3.3 A lower bound for $er(\mathbb{E}_1)$

The embedding rank $er(\mathbb{E}_1)$ of the near hexagon $\mathbb{E}_1$ is known to be equal to 24, see Brouwer et al. [4, p. 350] or Yoshiara [16, Theorem 1]. The fact that $er(\mathbb{E}_1) = 24$ was established in [4] with the aid of a computer and in [16] with some group theoretical argument involving the Leech lattice. In the present paper we determine $er(\mathbb{E}_1)$ in an entirely geometric way. The aim of this subsection is already to show that $er(\mathbb{E}_1) \geq 24$.

The technique we will use to prove this is more or less standard (see e.g. [6, Section 5]). For the calculation of $er(\mathbb{E}_1)$ we need to calculate the $\mathbb{F}_2$-rank of a certain matrix $N$, which is very hard without a computer. The $\mathbb{R}$-rank of $N$ is however easy to compute. This provides an upper bound for the $\mathbb{F}_2$-rank of $N$ and a lower bound for $er(\mathbb{E}_1)$. We explain this method in detail.

There are standard techniques for calculating the eigenvalues (and corresponding multiplicities) of the collinearity graph $\Gamma$ of $\mathbb{E}_1$, see [5] or [9, Section 3.3]. By [5, p. 427], the eigenvalues of $\Gamma$ are 24 (with multiplicity 1), 6 (with multiplicity 264), $-3$ (with multiplicity 440) and $-12$ (with multiplicity 24). Let $A$ be the collinearity matrix of $\mathbb{E}_1$, i.e. the adjacency matrix of $\Gamma$. The rows and columns of $A$ are indexed by the points of $\mathbb{E}_1$, where we use the same ordering $p_1, p_2, \ldots, p_{729}$ of the points. The $(i,j)$-th entry of $A$ is equal to 1 if $d(p_i, p_j) = 1$ and equal 0 otherwise. Let $N$ denote the incidence matrix of $\mathbb{E}_1$. The rows of $N$ are indexed by the points of $\mathbb{E}_1$ (same ordering as before) and the columns of $N$ are indexed by the lines of $\mathbb{E}_1$, with respect to a certain ordering $L_1, L_2, \ldots, L_{2916}$ of the lines. The $(i,j)$-th entry of $N$ is equal to 1 if $p_i \in L_j$ and equal to 0 otherwise. We have

$$N \cdot N^T = 12 \cdot I_{729} + A,$$

where $I_{729}$ is the $(729 \times 729)$-identity matrix. By the explicit construction of the universal embedding given in Section 2.2, we have

$$er(\mathbb{E}_1) = 729 - rank_{\mathbb{F}_2}(N).$$

Since the multiplicity of the eigenvalue $-12$ of $A$ is equal to 24, $rank_{\mathbb{F}_2}(N) \leq rank_{\mathbb{R}}(N) = rank_{\mathbb{R}}(NN^T) = rank(12 \cdot I_{729} + A) = 729 - 24 = 705$. It follows that

$$er(\mathbb{E}_1) = 729 - rank_{\mathbb{F}_2}(N) \geq 24.$$
Proof. Put $\text{PG}(5, 3) = \text{PG}(V)$, where $V$ is some 6-dimensional vector space over $\mathbb{F}_3$. We can choose a basis $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_6\}$ of $V$ such that $\text{PG}(4, 3) = \langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle$ and $X = \{(\bar{e}_1), (\bar{e}_2), (\bar{e}_3), (\bar{e}_4), (\bar{e}_5), (\bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \bar{e}_4 + \bar{e}_5)\}$. We will denote the point $\langle X_1 \bar{e}_1 + X_2 \bar{e}_2 + \cdots + X_6 \bar{e}_6 \rangle$ of $\text{PG}(5, 3)$ also by $(X_1, X_2, \ldots, X_6)$. Let $\alpha$ be the subspace of $\text{PG}(5, 3)$ generated by the points $(\bar{e}_1), (\bar{e}_2), (\bar{e}_3), (\bar{e}_4)$ and $(\bar{e}_6)$. Then $\alpha \setminus \text{PG}(4, 3)$ is a subspace of $\mathcal{A}^\ast$. The geometry induced on the set $\alpha \setminus \text{PG}(4, 3)$ is isomorphic to the Hamming near octagon $H(4, 3)$. By Corollary 3.4, there exists a set $Y_1$ of 16 points of the form $(\ast, \ast, \ast, \ast, 0, 1)$ such that $\alpha \setminus \text{PG}(4, 3)$ such that $Y_1 = \alpha \setminus \text{PG}(4, 3)$. So, $[Y_1]$ consists of all points of the form $(\ast, \ast, \ast, \ast, 0, 1)$. Now, put $Y_2 = \{(0, 0, 0, 0, 1, 1), (1, 0, 0, 0, 1, 1), (0, 0, 1, 0, 1, 1), (1, 0, 1, 0, 1, 1), (0, 1, 0, 0, 1, 1), (1, 1, 0, 1, 1)\}$. We prove that $Y := Y_1 \cup Y_2$ generates $\mathcal{A}^\ast$.

Since the collinear points $(0, 0, 0, 0, 1, 1)$ and $(1, 0, 0, 0, 1, 1)$ belong to $[Y]$, every point of the form $(\ast, 0, 0, 0, 1, 1)$ also belongs to $[Y]$. Since the collinear points $(0, 0, 1, 0, 1, 1)$ and $(1, 0, 1, 0, 1, 1)$ belong to $[Y]$, every point of the form $(\ast, 0, 1, 0, 1, 1)$ also belongs to $[Y]$. Since the collinear points $(0, 1, 0, 0, 1, 1)$ and $(1, 1, 0, 0, 1, 1)$ belong to $[Y]$, every point of the form $(\ast, 1, 0, 0, 1, 1)$ also belongs to $[Y]$. Since all points of the form $(\ast, 0, 0, 0, 1, 1)$ and $(\ast, 0, 1, 0, 1, 1)$ belong to $[Y]$, also all points of the form $(\ast, 0, 0, 0, 1, 1)$ belong to $[Y]$. Summarizing, we have:

(I) all points of the form $(\ast, \ast, 0, 0, 1, 1)$ belong to $[Y]$;

(II) all points of the form $(\ast, 0, \ast, 0, 1, 1)$ belong to $[Y]$.

Let $x_1, x_2 \in \mathbb{F}_3$. Since the collinear points $(x_1, x_2, 0, 0, 1, 1)$ and $(x_1 - 1, x_2 - 1, -1, -1, 0, 1)$ belong to $[Y]$, also $(x_1 + 1, x_2 + 1, 1, 1, -1, 1)$ belongs to $[Y]$. Since the collinear points $(x_1, 0, x_2, 0, 1, 1)$ and $(x_1 - 1, -1, x_2 - 1, -1, 0, 1)$ belong to $[Y]$, also $(x_1 + 1, 1, x_2 + 1, 1, -1, 1)$ belongs to $[Y]$. Summarizing, we have:

(III) all points of the form $(\ast, 1, 1, 1, -1, 1)$ belong to $[Y]$;

(IV) all points of the form $(\ast, 1, \ast, 1, -1, 1)$ belong to $[Y]$.

We will now make an observation. Suppose a certain point $(a, b, c, d, e, 1)$ belongs to $[Y]$, where $e \neq 0$. Since $(a, b, c, d, e, 1)$ is collinear with $(a, b, c, d, 0, 1) \in [Y]$, every point of the form $(a, b, c, d, \ast, 1)$ belongs to $[Y]$. Applying this observation to (I), (II), (III) and (IV) above, we find:

(I') all points of the form $(\ast, \ast, 0, 0, \ast, 1)$ belong to $[Y]$;

(II') all points of the form $(\ast, 0, \ast, 0, \ast, 1)$ belong to $[Y]$;

(III') all points of the form $(\ast, \ast, 1, 1, \ast, 1)$ belong to $[Y]$;

(IV') all points of the form $(\ast, 1, \ast, 1, \ast, 1)$ belong to $[Y]$.

Let $x_1, x_2 \in \mathbb{F}_3$. Since the collinear points $(x_1, x_2, 0, 0, -1, 1)$ and $(x_1 + 1, x_2 + 1, 1, 1, 0, 1)$ belong to $[Y]$, also $(x_1 - 1, x_2 - 1, -1, -1, 1)$ belongs to $[Y]$. By the above observation, also $(x_1 - 1, x_2 - 1, -1, -1, -1, 1) \in [Y]$. By (I') and (III'), we can now conclude that:

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(I") all points of the form \((*,*,k,k,*,1)\) belong to \([Y]\).

Let \(x_1, x_2 \in \mathbb{F}_3\). Since the collinear points \((x_1, 0, x_2, 0, -1, 1)\) and \((x_1 + 1, 1, x_2 + 1, 1, 0, 1)\) belong to \([Y]\), also \((x_1 - 1, -1, x_2 - 1, -1, 1, 1)\) belongs to \([Y]\). By the above observation, also \((x_1 - 1, -1, x_2 - 1, -1, -1, 1)\) belongs to \([Y]\). By (II') and (IV'), we can now conclude that:

(II") all points of the form \((*,k,*,k,*,1)\) belong to \([Y]\).

We now prove that every point of the form \((a,b,c,d,e,1)\), \(b \neq c\), belongs to \([Y]\). By (I'), this is true if \(c = d\). By (II'), this is true if \(b = d\). So, in the sequel, we may suppose that \(b, c\) and \(d\) are mutually distinct. Since the collinear points \((a,b,c,c,e,1)\) and \((a,b,c,b,e,1)\) belong to \([Y]\), also \((a,b,c,d,e,1)\) must belong to \([Y]\).

We now also prove that every point of the form \((a,b,b,d,e,1)\) belongs to \([Y]\). But this follows from the fact that the collinear points \((a,b,b+1,d,e,1)\) and \((a,b,b-1,d,e,1)\) belong to \([Y]\).

Lemma 4.2 Let \(Y\) be a subset of size \(i \in \{1,2,3,4\}\) of \(X\), let \(\alpha\) be an \(i\)-dimensional subspace of \(PG(5,3)\) which intersects \(PG(4,3)\) in the subspace \(\langle Y \rangle\) and put \(S := \alpha \setminus PG(4,3)\). Then every generating set of size \(2^i\) of \(\tilde{S} \cong H(i,3)\) can be extended to a generating set of size 22 of \(A^*\).

Proof. As in the proof of Lemma 4.1, put \(PG(5,3) = PG(V)\) where \(V\) is a 6-dimensional vector space over \(\mathbb{F}_3\) and choose a basis \(\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_6\}\) of \(V\) such that \(PG(4,3) = \langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle\), \(X = \{(\bar{e}_1), (\bar{e}_2), (\bar{e}_3), (\bar{e}_4), (\bar{e}_5), (\bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \bar{e}_4 + \bar{e}_5)\}\), \(\langle \bar{e}_6 \rangle \in \alpha\) and \(Y = \{(\bar{e}_1), (\bar{e}_2), (\bar{e}_3), (\bar{e}_4), (\bar{e}_5)\}\). Let \(\beta\) be the subspace of \(PG(5,3)\) generated by the points \(\bar{e}_1\), \(\bar{e}_2\), \(\bar{e}_3\), \(\bar{e}_4\) and \(\bar{e}_6\). Then \(S' := \beta \setminus PG(4,3)\) is a subspace of \(A^*\) and \(\tilde{S}' \cong H(4,3)\). Notice that \(\tilde{S}\) is a subgeometry of \(\tilde{S}'\). By Lemma 3.3, every generating set of size \(2^i\) of \(\tilde{S}\) can be extended to a generating set \(Y_1\) of size 16 of \(\tilde{S}'\). Now, by the proof of Lemma 4.1, \(Y_1\) can be extended to a generating set of size 22 of \(A^*\).

5 Proofs of the main theorems

In this section, we prove Theorems 1.1, 1.2 and 1.3. Let \(\Pi_\infty\) be a hyperplane of the projective space \(PG(6,3)\). Let \(K^*\) be the set of 12 points of \(\Pi_\infty\) as defined in Section 2.1. Then the point-line geometry \(T_5^*(K^*)\) is isomorphic to the near hexagon \(E_1\). Recall that by Theorems 3.1 and 3.2 we know all subgeometries of \(T_5^*(K^*)\) which are either special or isomorphic to \(H(i,3)\) for some \(i \in \{1,2,3,4\}\).

Lemma 5.1 Let \(A\) be a special subgeometry of \(T_5^*(K^*)\). Then any generating set \(Y\) of size 22 of \(A\) can be extended to a generating set of size 24 of \(T_5^*(K^*)\).

Proof. Let \(\alpha\) be the subspace of \(PG(6,3)\) generated by the points of \(A\). Let \(x_1, x_2, \ldots, x_6\) denote the points of \(K^*\) contained in \(\alpha\) and let \(x_7, x_8, \ldots, x_{12}\) denote the remaining 6 points of \(K^*\). Then \(\langle x_1, x_2, \ldots, x_6 \rangle\) and \(\langle x_7, x_8, \ldots, x_{12} \rangle\) are two hyperplanes of \(\Pi_\infty\). Let \(y_1\) and \(y_2\) be two points of \(T_5^*(K^*)\) outside \(\alpha\) such that \(\beta_1 := \langle x_7, x_8, \ldots, x_{12}, y_1 \rangle\) and \(\beta_2 :=\)
\[ \langle x_7, x_8, \ldots, x_{12}, y_2 \rangle \] are two distinct hyperplanes of \( \text{PG}(6, 3) \) through \( \langle x_7, x_8, \ldots, x_{12} \rangle \). We prove that \( Y \cup \{y_1, y_2\} \) is a generating set of \( T_5^* (K^*) \).

Let \( i \in \{1, 2\} \). We first prove that every point of \( \beta_i \setminus \Pi_{\infty} \) is contained in \( [Y \cup \{y_i\}] \). The set \( \beta_i \setminus \Pi_{\infty} \) is the point-set of a subgeometry \( A_i \cong H(5, 3) \) of \( T_5^* (K^*) \) whose lines are those lines of \( T_5^* (K^*) \) contained in \( \beta_i \) whose points at infinity belong to the set \( \{x_7, x_8, x_9, x_{10}, x_{11}\} \). Notice that \( [Y] = \alpha \setminus \Pi_{\infty} \). Since \( \alpha \cap \beta_i \) is a hyperplane of \( \beta_i \), \( O_i := [Y] \cap (\beta_i \setminus \Pi_{\infty}) \) is an ovoid of \( A_i \). Now, the complement of any ovoid of \( H(5, 3) \) is connected by Blok and Brouwer [2, Theorem 7.3] or Shult [14, Lemma 6.1]. Since \( y_i \notin O_i \), we have \( \beta_i \setminus \Pi_{\infty} \subseteq [Y \cup \{y_i\}] \).

Summarizing, we have \( (\alpha \cup \beta_1 \cup \beta_2) \setminus \Pi_{\infty} \subseteq [Y \cup \{y_1, y_2\}] \). Now, let \( z \) be an arbitrary point of \( T_5^* (K^*) \). Since the line \( x_1z \) of \( T_5^* (K^*) \) contains two distinct points of \( (\alpha \cup \beta_1 \cup \beta_2) \setminus \Pi_{\infty} \), namely the unique points in \( x_1z \cap \beta_1 \) and \( x_1z \cap \beta_2 \), we have \( z \in [Y \cup \{y_1, y_2\}] \). Since \( z \) was an arbitrary point of \( T_5^* (K^*) \), \( Y \cup \{y_1, y_2\} \) is a generating set of \( T_5^* (K^*) \).

**Proposition 5.2** The embedding and generating ranks of \( T_5^* (K^*) \) are equal to 24.

**Proof.** By Lemmas 4.1 and 5.1 we have \( gr(T_5^* (K^*)) \leq 24 \) and by Section 3.3 we know that \( er(T_5^* (K^*)) \geq 24 \). Since \( er(T_5^* (K^*)) \leq gr(T_5^* (K^*)) \), we have \( er(T_5^* (K^*)) = gr(T_5^* (K^*)) = 24 \).

**Proposition 5.3** Let \( A \) be a special subgeometry of \( T_5^* (K^*) \). Then the embedding and generating ranks of \( A \) are equal to 22. Moreover, the projective embedding of \( A \) induced by the universal embedding of \( T_5^* (K^*) \) is isomorphic to the universal embedding of \( A \).

**Proof.** Let \( \tilde{\Sigma} \) denote the projective space which affords the universal embedding \( \tilde{e} \) of \( T_5^* (K^*) \). By Lemmas 4.1 and 5.1, there exists a generating set \( \{x_1, x_2, \ldots, x_{24}\} \) of size 24 of \( T_5^* (K^*) \) such that \( \{x_1, x_2, \ldots, x_{24}\} \) is a generating set of \( A \). Since \( er(A) = 22 \), the points \( \tilde{e}(x_1), \tilde{e}(x_2), \ldots, \tilde{e}(x_{24}) \) of \( \tilde{\Sigma} \) are linearly independent. So, the embedding \( \tilde{e} \) induces an embedding \( e \) of \( A \) into the 21-dimensional subspace \( \langle \tilde{e}(x_1), \tilde{e}(x_2), \ldots, \tilde{e}(x_{24}) \rangle \) of \( \tilde{\Sigma} \). It follows that \( er(A) \geq 22 \). By Lemma 4.1, \( gr(A) \leq 22 \). Since \( er(A) \leq gr(A) \), we must have \( er(A) = gr(A) = 22 \). Hence, the embedding \( e \) must be universal.

**Lemma 5.4** Let \( i \in \{1, 2, 3, 4\} \) and let \( A \) be a subgeometry of \( T_5^* (K^*) \) isomorphic to \( H(i, 3) \). Then any generating set of size 24 of \( A \) can be extended to a generating set of size 24 of \( T_5^* (K^*) \).

**Proof.** This is a corollary of Lemmas 4.2 and 5.1.

**Proposition 5.5** Let \( i \in \{1, 2, 3, 4\} \) and let \( A \) be a subgeometry of \( T_5^* (K^*) \) isomorphic to \( H(i, 3) \). Then the projective embedding of \( A \) induced by the universal embedding of \( T_5^* (K^*) \) is isomorphic to the universal embedding of \( A \).

**Proof.** Let \( \tilde{\Sigma} \) denote the projective space which affords the universal embedding \( \tilde{e} \) of \( T_5^* (K^*) \). By Corollary 3.4 and Lemma 5.4, there exists a generating set \( \{x_1, x_2, \ldots, x_{24}\} \) of size 24 of \( T_5^* (K^*) \) such that \( \{x_1, x_2, \ldots, x_{24}\} \) is a generating set of \( A \). Since \( er(T_5^* (K^*)) = 22 \), the
24, the points \( \tilde{e}(x_1), \tilde{e}(x_2), \ldots, \tilde{e}(x_{24}) \) of \( \tilde{\Sigma} \) are linearly independent. It follows that \( \tilde{e} \) induces an embedding \( e \) of \( \mathcal{A} \) into the \((2^r - 1)\)-dimensional subspace \( \langle \tilde{e}(x_1), \tilde{e}(x_2), \ldots, \tilde{e}(x_{24}) \rangle \) of \( \tilde{\Sigma} \). By Proposition 3.5(2), \( e \) must be isomorphic to the universal embedding of \( \mathcal{A} \).

**Lemma 5.6** Let \( S \) be a subspace of \( T_5^* (\mathcal{K}^*) \) such that \( \tilde{S} \) is either a special subgeometry or a subgeometry isomorphic to \( H(i, 3) \) for some \( i \in \{1, 2, 3, 4\} \). Then \( S \) is the intersection of a number of hyperplanes of \( T_5^* (\mathcal{K}^*) \).

**Proof.** (1) First, suppose that \( S \) is a subspace of \( T_5^* (\mathcal{K}^*) \) such that \( \tilde{S} \) is a special subgeometry of \( T_5^* (\mathcal{K}^*) \). Let \( \alpha \) be the subspace of \( \text{PG}(6, 3) \) generated by all points of \( S \). Then \( \alpha \cap \mathcal{K}^* \) is a set of 6 points, say \( \alpha \cap \mathcal{K}^* = \{x_1, x_2, \ldots, x_6\} \). Put \( \{x_7, x_8, \ldots, x_{12}\} = \mathcal{K}^* \setminus \{x_1, x_2, \ldots, x_6\} \). Now, \( \langle x_1, x_2, \ldots, x_6 \rangle \) and \( \langle x_7, x_8, \ldots, x_{12} \rangle \) are two hyperplanes of \( \Pi_\infty \).

Let \( \beta_1 \) and \( \beta_2 \) be two distinct hyperplanes of \( \text{PG}(6, 3) \) through \( \langle x_7, x_8, \ldots, x_{12} \rangle \) distinct from \( \Pi_\infty \). Put \( S_i := \beta_i \setminus \Pi_\infty, i \in \{1, 2\} \). It is easily seen that every line of \( T_5^* (\mathcal{K}^*) \) is either contained in \( S_1 \cup S \) or intersects \( S_1 \cup S \) in a unique point. Hence, \( S_1 \cup S \) is a hyperplane of \( T_5^* (\mathcal{K}^*) \).

Clearly, \( S = (S_1 \cup S) \cap (S_2 \cup S) \).

(2) Let \( i \in \{1, 2, 3, 4\} \) and let \( S \) be a subspace of \( T_5^* (\mathcal{K}^*) \) such that \( \tilde{S} \) is a subgeometry of \( T_5^* (\mathcal{K}^*) \) isomorphic to \( H(i, 3) \). If \( \mathcal{F} \) denotes the set of all special subgeometries of \( T_5^* (\mathcal{K}^*) \) containing all points of \( S \), then using Theorems 3.1 and 3.2 it can readily be verified that \( S = \bigcap_{F \in \mathcal{F}} P_F \), where \( P_F \) denotes the point set of \( F \in \mathcal{F} \). Since each \( P_F, F \in \mathcal{F}, \) is the intersection of two hyperplanes, \( S \) is the intersection of a number of hyperplanes of \( T_5^* (\mathcal{K}^*) \).

**Proposition 5.7** Let \( S \) be a subspace of \( T_5^* (\mathcal{K}^*) \) such that \( \tilde{S} \) is either a special subgeometry or a subgeometry isomorphic to \( H(i, 3) \) for some \( i \in \{1, 2, 3, 4\} \). Let \( \tilde{e} \) denote the universal embedding of \( T_5^* (\mathcal{K}^*) \). If \( x \) is a point of \( T_5^* (\mathcal{K}^*) \), then \( x \in S \) if and only if \( \tilde{e}(x) \in \langle \tilde{e}(S) \rangle \).

**Proof.** Let \( H_1, H_2, \ldots, H_k \) be \( k \geq 2 \) hyperplanes of \( T_5^* (\mathcal{K}^*) \) such that \( S = H_1 \cap H_2 \cap \cdots \cap H_k \), let \( \mathcal{P} \) denote the point set of \( T_5^* (\mathcal{K}^*) \) and let \( \Sigma = \text{PG}(23, 2) \) be the projective space which affords the universal embedding of \( T_5^* (\mathcal{K}^*) \). Recall that by Ronan [13], the hyperplane \( H_i, i \in \{1, 2, \ldots, k\} \), arises from \( \tilde{e} \), i.e. there is a hyperplane \( \Pi_i \) of \( \tilde{\Sigma} \) such that \( \tilde{e}(H_i) = \tilde{e}(\mathcal{P}) \cap \Pi_i \). So, \( \tilde{e}(S) = \tilde{e}(H_1 \cap H_2 \cap \cdots \cap H_k) = \tilde{e}(H_1) \cap \tilde{e}(H_2) \cap \cdots \cap \tilde{e}(H_k) = \tilde{e}(\mathcal{P}) \cap (\Pi_1 \cap \Pi_2 \cap \cdots \cap \Pi_k) \). Hence, \( \langle \tilde{e}(S) \rangle \subseteq \Pi_1 \cap \Pi_2 \cap \cdots \cap \Pi_k \). Since \( \tilde{e}(S) \subseteq \langle \tilde{e}(S) \rangle \cap \tilde{e}(\mathcal{P}) \subseteq \Pi_1 \cap \Pi_2 \cap \cdots \cap \Pi_k \cap \tilde{e}(\mathcal{P}) = \tilde{e}(S) \), we have \( \tilde{e}(S) = \langle \tilde{e}(S) \rangle \cap \tilde{e}(\mathcal{P}) \).

This is precisely what we needed to prove.

**References**


