Implication functions in interval-valued fuzzy set theory

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Abstract Interval-valued fuzzy set theory is an extension of fuzzy set theory in which the real, but unknown, membership degree is approximated by a closed interval of possible membership degrees. Since implications on the unit interval play an important role in fuzzy set theory, several authors have extended this notion to interval-valued fuzzy set theory. This chapter gives an overview of the results pertaining to implications in interval-valued fuzzy set theory. In particular, we describe several possibilities to represent such implications using implications on the unit interval, we give a characterization of the implications in interval-valued fuzzy set theory which satisfy the Smets-Magrez axioms, we discuss the solutions of a particular distributivity equation involving strict t-norms, we extend monoidal logic to the interval-valued fuzzy case and we give a soundness and completeness theorem which is similar to the one existing for monoidal logic, and finally we discuss some other constructions of implications in interval-valued fuzzy set theory.

1 Introduction

Fuzzy set theory has been introduced by Zadeh [57] in order to deal with the imprecision, ignorance and vagueness present in the real world, and has been applied successfully in several areas. In fuzzy set theory the membership of an object in a set is determined by assigning a real number between 0 and 1, called the membership degree of the object in the set. However, in some real problems, it is very difficult to determine a correct value (if there is one) for the membership degrees. In many cases only an approximated value of the membership degree is given. This kind of uncertainty in the membership degrees has motivated several extensions of Zadeh's

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fuzzy set theory, such as Atanassov's intuitionistic fuzzy set theory [1], interval-valued fuzzy set theory [43], type-2 fuzzy set theory [58], ... Interval-valued fuzzy sets assign to each object instead of a single number a closed interval which approximates the real, but unknown, membership degree. As such, interval-valued fuzzy set theory forms a good balance between the ease of use of fuzzy set theory and the expressiveness of type-2 fuzzy set theory. Since the underlying lattice of Atanassov's intuitionistic fuzzy set theory is isomorphic to the underlying lattice of interval-valued fuzzy set theory, any results about any functions on any of those lattices hold for both theories. Therefore we will focus in this work to functions defined on the underlying lattice of interval-valued fuzzy set theory. Interval-valued fuzzy sets and Atanassov's intuitionistic fuzzy sets have been investigated both theoretically and practically by many researchers [12, 13, 15, 33, 38, 40, 45, 53, 54, 55, 60].

Since implications on the unit interval play an important role in fuzzy set theory [9], several authors have extended this notion to interval-valued fuzzy set theory [4, 6, 7, 8, 11, 17, 26, 48]. This chapter gives an overview of the results pertaining to implications in interval-valued fuzzy set theory. In the next section we start with some preliminary definitions concerning the underlying structure of interval-valued fuzzy set theory and some functions which we will need later on. This section is followed by several sections in which we give an overview of known results.

2 Preliminary Definitions

2.1 The Lattice $\mathcal{L}^I$

The underlying lattice $\mathcal{L}^I$ of interval-valued fuzzy set theory is given as follows.

Definition 1. We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where

$$L^I = \{[x_1,x_2] \mid (x_1,x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\},$$
$$[x_1,x_2] \leq_{L^I} [y_1,y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2), \text{ for all } [x_1,x_2],[y_1,y_2] \text{ in } L^I.$$

Similarly as Lemma 2.1 in [24] it is shown that $\mathcal{L}^I$ is a complete lattice.

Definition 2. [34, 43] An interval-valued fuzzy set on U is a mapping $A : U \to L^I$.

Definition 3. [1, 2, 3] An intuitionistic fuzzy set in the sense of Atanassov on U is a set

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

where $\mu_A(u) \in [0,1]$ denotes the membership degree and $\nu_A(u) \in [0,1]$ the non-membership degree of u in A and where for all $u \in U$, $\mu_A(u) + \nu_A(u) \leq 1$.

An intuitionistic fuzzy set in the sense of Atanassov A on U can be represented by the $\mathcal{L}^I$-fuzzy set A given by

$$\begin{array}{rcll} A : U & \to & L^I : & \\ u & \mapsto & [\mu_A(u), 1 - \nu_A(u)], & \forall u \in U. \end{array}$$

In Figure 1 the set L^I is shown. Note that to any element $x = [x_1, x_2]$ of L^I there corresponds a point $(x_1, x_2) \in \mathbb{R}^2$.

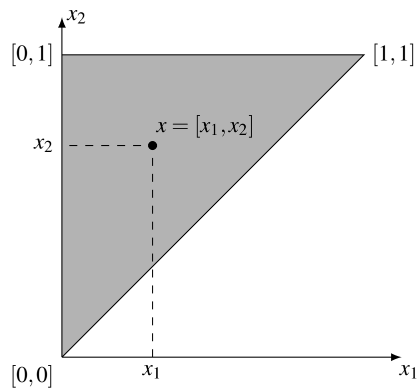

Fig. 1 The grey area is L^I.

In the sequel, if $x \in L^I$, then we denote its bounds by $x_1 = \mathrm{pr}_1(x)$ and $x_2 = \mathrm{pr}_2(x)$, i.e. $x = [x_1, x_2]$. The smallest and the largest element of $\mathscr{L}^I$ are given by $0_{\mathscr{L}^I} = [0,0]$ and $1_{\mathscr{L}^I} = [1,1]$. The hypothenuse of the triangle corresponds to the set $D = \{[x_1, x_1] \mid x_1 \in [0,1]\}$ of values in L^I about which there is no indeterminacy and can be identified with the unit interval $[0,1]$ from (classical) fuzzy set theory. The elements of D are called the *exact elements* of the lattice $\mathscr{L}^I$. Note that, for x, y in L^I, $x <_{L^I} y$ is equivalent to "$x \leq_{L^I} y$ and $x \neq y$", i.e. either $x_1 < y_1$ and $x_2 \leq y_2$, or $x_1 \leq y_1$ and $x_2 < y_2$. We denote by $x \ll_{L^I} y$: $x_1 < y_1$ and $x_2 < y_2$.

Bedregal et al. [12, 44] introduced the notion of *interval representation*, where an interval function $F : L^I \to L^I$ represents a real function $f : [0,1] \to [0,1]$ if for each $X \in L^I$, $f(x) \in F(X)$ whenever $x \in X$ (the interval X represents the real x). So, F is an interval representation of f if $F(X)$ includes all possible situations that could occur if the uncertainty in X were to be expelled. For $f : [0,1] \to [0,1]$, the function $\hat{f} : L^I \to L^I$ defined by

$$\hat{f}(X) = [\inf\{f(x) \mid x \in X\}, \sup\{f(x) \mid x \in X\}]$$

is an interval representation of f [12, 44]. Clearly, if F is also an interval representation of f, then for each $X \in L^I$, $\hat{f}(X) \subseteq F(X)$. Thus, $\hat{f}$ returns a narrower interval than any other interval representation of f and is therefore its *best interval representation*.

2.2 Triangular Norms, Implications and Negations on $\mathscr{L}^I$

Implications are often generated from other connectives. In this section we will introduce some of these connectives and give the construction of implications derived from these functions.

Definition 4. A t-norm on a complete lattice $\mathscr{L} = (L, \leq_L)$ is a commutative, associative, increasing mapping $\mathscr{T} : L^2 \to L$ which satisfies $\mathscr{T}(1_{\mathscr{L}}, x) = x$, for all $x \in L$.

A t-conorm on a complete lattice $\mathscr{L} = (L, \leq_L)$ is a commutative, associative, increasing mapping $\mathscr{S} : L^2 \to L$ which satisfies $\mathscr{S}(0_{\mathscr{L}}, x) = x$, for all $x \in L$.

Let $\mathscr{T}$ be a t-norm on a complete lattice $\mathscr{L} = (L, \leq_L)$ and $x \in L$, then we denote $x^{(n)_{\mathscr{T}}} = \mathscr{T}(x, x^{(n-1)_{\mathscr{T}}})$, for $n \in \mathbb{N} \setminus \{0,1\}$, and $x^{(1)_{\mathscr{T}}} = x$.

Example 1. Some well-known t-(co)norms on $([0,1], \leq)$ are the Łukasiewicz t-norm T_L, the product t-norm T_P and the Łukasiewicz t-conorm defined by, for all x, y in $[0,1]$,

$$\begin{aligned} T_L(x,y) &= \max(0, x+y-1), \\ T_P(x,y) &= xy, \\ S_L(x,y) &= \min(1, x+y). \end{aligned}$$

For t-norms on $\mathscr{L}^I$, we consider the following special classes.

Lemma 1. *[21]*

- *Given t-norms T_1 and T_2 on $([0,1], \leq)$ with $T_1 \leq T_2$, the mapping $\mathscr{T}_{T_1,T_2} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathscr{T}_{T_1,T_2}(x,y) = [T_1(x_1,y_1), T_2(x_2,y_2)].$$

  is a t-norm on $\mathscr{L}^I$.
- *Given a t-norm T on $([0,1], \leq)$, the mappings $\mathscr{T}_T : (L^I)^2 \to L^I$ and $\mathscr{T}'_T : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\begin{aligned} \mathscr{T}_T(x,y) &= [T(x_1,y_1), \max(T(x_1,y_2), T(x_2,y_1))], \\ \mathscr{T}'_T(x,y) &= [\min(T(x_1,y_2), T(x_2,y_1)), T(x_2,y_2)], \end{aligned}$$

  are t-norms on $\mathscr{L}^I$.

Definition 5. [21] Let T_1, T_2 and T be t-norms on $([0,1], \leq)$. The t-norms $\mathscr{T}_{T_1,T_2}$, $\mathscr{T}_T$ and $\mathscr{T}'_T$ defined in Lemma 1 are called the t-representable t-norm on $\mathscr{L}^I$ with representatives T_1 and T_2, the optimistic t-norm and the pessimistic t-norm on $\mathscr{L}^I$ with representative T, respectively. In a similar way t-representable, pessimistic and optimistic t-conorms on $\mathscr{L}^I$ can be defined.

Note that $\mathscr{T}_{T,T}$ is the best interval representation of T. Furthermore, if T is continuous[1],

$$\mathscr{T}_{T,T}([x_1,x_2],[y_1,y_2]) = \{T(\alpha,\beta) \mid \alpha \in [x_1,x_2] \text{ and } \beta \in [y_1,y_2]\}. \tag{1}$$

Looking at the structure of $\mathscr{T}_T$, this t-norm has the same lower bound as the t-representable t-norm $\mathscr{T}_{T,T}$, but differs from it by its upper bound: instead of taking the "optimum" value $T(x_2,y_2)$, the second component is obtained by taking the maximum of $T(x_1,y_2)$ and $T(x_2,y_1)$. Hence it is not guaranteed that the interval $\mathscr{T}_T(x,y)$ contains all possible values $T(\alpha,\beta)$ for $\alpha \in [x_1,x_2]$ and $\beta \in [y_1,y_2]$. Rather (for continuous T),

$$\mathscr{T}_T([x_1,x_2],[y_1,y_2]) = \{T(\alpha,y_1) \mid \alpha \in [x_1,x_2]\} \cup \{T(x_1,\beta) \mid \beta \in [y_1,y_2]\}. \tag{2}$$

What this representation enforces is that, in eliminating the uncertainty from x and y, we have to impose for at least one of them the "worst" possible value (x_1, resp. y_1). Therefore, this could be called a *pessimistic* approach to the definition of a t-norm on $\mathscr{L}^I$, hence the name "pessimistic t-norm". Similarly, the adapted upper bound of $\mathscr{T}'_T$ reflects an optimistic approach.

A class of t-norms generalizing both the t-representable t-norms and the pessimistic t-norms can be introduced. Let T be a t-norm on $([0,1],\leq)$, and $t \in [0,1]$. Then the mapping $\mathscr{T}_{T,t} : (L^I)^2 \to L^I$ defined by, for all x,y in L^I,

$$\mathscr{T}_{T,t}(x,y) = [T(x_1,y_1), \max(T(t,T(x_2,y_2)), T(x_1,y_2), T(x_2,y_1))],$$

is a t-norm on $\mathscr{L}^I$ [23]. The usage of this class is that it allows the user to define $\mathscr{T}([0,1],[0,1]) = [0,t]$ arbitrarily. This can be useful in applications where in some situations one needs to impose that the conjunction of two completely unknown propositions is also unknown (e.g. "the sun will shine tomorrow" and "this night it will freeze"), while in other situations it would be more appropriate that the conjunction of two unknown statements is false (e.g. "this night it will freeze" and "this night it will be hot"). If $t = 0$, then we obtain the pessimistic t-norms, if $t = 1$, then we find t-representable t-norms. Clearly, since the lower bound of $\mathscr{T}_{T,t}(x,y)$ is independent of x_2 and y_2, the optimistic t-norms do not belong to this class as soon as $T \neq \min$.

Definition 6. An implication on a complete lattice $\mathscr{L} = (L,\leq_L)$ is a mapping $\mathscr{I} : L^2 \to L$ that is decreasing (w.r.t. $\leq_L$) in its first, and increasing (w.r.t. $\leq_L$) in its second argument, and that satisfies

$$\mathscr{I}(0_\mathscr{L},0_\mathscr{L}) = 1_\mathscr{L}, \quad \mathscr{I}(0_\mathscr{L},1_\mathscr{L}) = 1_\mathscr{L},$$
$$\mathscr{I}(1_\mathscr{L},1_\mathscr{L}) = 1_\mathscr{L}, \quad \mathscr{I}(1_\mathscr{L},0_\mathscr{L}) = 0_\mathscr{L}.$$

[1] The continuity is necessary in order to have an equality in (1). In the general case it only holds that the left-hand side of (1) is a subset of the right-hand side.

Definition 7. A negation on a complete lattice $\mathscr{L} = (L, \leq_L)$ is a decreasing mapping $\mathscr{N} : L \to L$ for which $\mathscr{N}(0_{\mathscr{L}}) = 1_{\mathscr{L}}$ and $\mathscr{N}(1_{\mathscr{L}}) = 0_{\mathscr{L}}$. If $\mathscr{N}(\mathscr{N}(x)) = x$, for all $x \in L$, then $\mathscr{N}$ is called involutive.

Proposition 1. *[6] Let $IVFI$ be the set of all implications on $\mathscr{L}^I$. Then $(IVFI, \inf, \sup)$ is a complete lattice, i.e.*

$$(\forall t \in T)(\mathscr{I}_t \in IVFI) \implies (\sup_{t \in T} \mathscr{I}_t, \inf_{t \in T} \mathscr{I}_t) \in IVFI^2.$$

Corollary 1. *[6] $(IVFI, \inf, \sup)$ has the greatest element*

$$\mathscr{I}_1(x,y) = \begin{cases} 0_{\mathscr{L}^I}, & \text{if } x = 1_{\mathscr{L}^I} \text{ and } y = 0_{\mathscr{L}^I}, \\ 1_{\mathscr{L}^I}, & \text{otherwise}, \end{cases}$$

and the least element

$$\mathscr{I}_0(x,y) = \begin{cases} 1_{\mathscr{L}^I}, & \text{if } x = 0_{\mathscr{L}^I} \text{ or } y = 1_{\mathscr{L}^I}, \\ 0_{\mathscr{L}^I}, & \text{otherwise}. \end{cases}$$

Implications are often derived from other types of connectives. For our purposes, we consider S- and R-implications:

- let $\mathscr{T}$ be a t-norm on $\mathscr{L}$, then the residual implication or R-implication $\mathscr{I}_{\mathscr{T}}$ is defined by, for all x, y in L,

$$\mathscr{I}_{\mathscr{T}}(x,y) = \sup\{z \mid z \in L \text{ and } \mathscr{T}(x,z) \leq_L y\}; \tag{3}$$

- let $\mathscr{S}$ be a t-conorm and $\mathscr{N}$ a negation on $\mathscr{L}$, then the S-implication $\mathscr{I}_{\mathscr{S},\mathscr{N}}$ is defined by, for all x, y in L,

$$\mathscr{I}_{\mathscr{S},\mathscr{N}}(x,y) = \mathscr{S}(\mathscr{N}(x), y). \tag{4}$$

We say that a t-norm $\mathscr{T}$ on $\mathscr{L}$ satisfies the *residuation principle* if and only if, for all x, y, z in L,

$$\mathscr{T}(x,y) \leq_L z \iff y \leq_L \mathscr{I}_{\mathscr{T}}(x,z).$$

Example 2. The residual implications of the t-norms given in Example 1 are given by, for all x, y in $[0,1]$,

$$I_{T_L}(x,y) = \min(1, y + 1 - x),$$
$$I_{T_P}(x,y) = \min\left(1, \frac{y}{x}\right),$$

using the convention $\frac{y}{x} = +\infty$, for $x = 0$ and $y \in [0,1]$.

Example 3. Using the Łukasiewicz t-norm and t-conorm given in Example 1 the following t-norm, t-conorm and implication on $\mathscr{L}^I$ can be constructed. For all x, y in L^I,

$$\mathscr{T}_{T_L}(x,y) = [\max(0, x_1 + y_1 - 1), \max(0, x_1 + y_2 - 1, x_2 + y_1 - 1)],$$
$$\mathscr{S}_{S_L}(x,y) = [\min(1, x_1 + y_2, x_2 + y_1), \min(1, x_2 + y_2)],$$
$$\mathscr{I}_{\mathscr{T}_{T_L}}(x,y) = \mathscr{I}_{\mathscr{S}_{S_L},\mathscr{N}_s}(x,y) = [\min(1, y_1 + 1 - x_1, y_2 + 1 - x_2), \min(1, y_2 + 1 - x_1)],$$

where $\mathscr{N}_s$ is the standard negation on $\mathscr{L}^I$ defined by $\mathscr{N}_s([x_1,x_2]) = [1-x_2, 1-x_1]$, for all $[x_1,x_2] \in L^I$.

Example 4. [23] Let T be an arbitrary t-norm on $([0,1],\leq)$ and $t \in [0,1]$. The residual implication $\mathscr{I}_{\mathscr{T}_{T,t}}$ of $\mathscr{T}_{T,t}$ is given by, for all x, y in L^I,

$$\mathscr{I}_{\mathscr{T}_{T,t}}(x,y) = [\min(I_T(x_1,y_1), I_T(x_2,y_2)), \min(I_T(T(x_2,t),y_2), I_T(x_1,y_2))].$$

Proposition 2. [22] *Let $\mathscr{N}$ be a negation on $\mathscr{L}^I$. Then $\mathscr{N}$ is involutive if and only if there exists an involutive negation N on $([0,1],\leq)$ such that, for all $x \in L^I$,*

$$\mathscr{N}(x) = [N(x_2), N(x_1)].$$

Definition 8. For any negation $\mathscr{N}$ on $\mathscr{L}^I$, if there exists negations N_1 and N_2 on $([0,1],\leq)$ with $N_1 \leq N_2$ such that $\mathscr{N}(x) = [N_1(x_2), N_2(x_1)]$, for all $x \in L^I$, then we denote $\mathscr{N}$ by $\mathscr{N}_{N_1,N_2}$, we call $\mathscr{N}$ n-representable and we call N_1 and N_2 the representatives of $\mathscr{N}$.

Note that $\mathscr{N}_{N,N}$ is the best interval representation of N. Furthermore, if N is continuous, then

$$\mathscr{N}_{N,N}([x_1,x_2]) = \{N(x) \mid x \in [x_1,x_2]\}.$$

Proposition 3. [19, 22] *A mapping $\Phi : L^I \to L^I$ is an increasing permutation of $\mathscr{L}^I$ with increasing inverse if and only if there exists an increasing permutation ϕ of $([0,1],\leq)$ such that, for all $x \in L^I$,*

$$\Phi(x) = [\phi(x_1), \phi(x_2)].$$

Let $n \in \mathbb{N} \setminus \{0\}$. If for an n-ary mapping f on $[0,1]$ and an n-ary mapping F on L^I it holds that $F([a_1,a_1],\ldots,[a_n,a_n]) = [f(a_1,\ldots,a_n), f(a_1,\ldots,a_n)]$, for all $(a_1,\ldots,a_n) \in [0,1]^n$, then we say that F is a natural extension of f to L^I. Clearly, for any mapping F on L^I, $F(D,\ldots,D) \subseteq D$ if and only if there exists a mapping f on $[0,1]$ such that F is a natural extension of f to L^I. E.g. for any t-norm T on $([0,1],\leq)$, the t-norms $\mathscr{T}_{T,T}$ and $\mathscr{T}_T$ are natural extensions of T to L^I; if $\mathscr{N}$ is an involutive negation on $\mathscr{L}^I$, then from Proposition 2 it follows that there exists an involutive negation N on $([0,1],\leq)$ such that $\mathscr{N}$ is a natural extension of N.

2.3 Continuity on $\mathscr{L}^I$

In order to introduce continuity on $\mathscr{L}^I$ we need a metric on L^I. Well-known metrics include the Euclidean distance, the Hamming distance and the Moore distance. In the two-dimensional space $\mathbb{R}^2$ they are defined as follows:

- the Euclidean distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ is given by
$$d^E(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\ ,$$
- the Hamming distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ is given by
$$d^H(x,y) = |x_1 - y_1| + |x_2 - y_2|,$$
- the Moore distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ is given by [39]
$$d^M(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|).$$

If we restrict these distances to L^I then we obtain the metric spaces (L^I, d^E), (L^I, d^H) and (L^I, d^M). Note that these distances are homeomorphic when used on $\mathbb{R}^2$ (see [14]). Therefore, the relative topologies w.r.t. L^I are also homeomorphic, which implies that they determine the same set of continuous functions. From now on, if we talk about continuity in L^I, then we mean continuity w.r.t. one of these metric spaces.

It is shown in [22] that for t-norms on $\mathscr{L}^I$ the residuation principle is *not* equivalent to the left-continuity and not even to the continuity of the t-norm: all t-norms on $\mathscr{L}^I$ which satisfy the residuation principle are left-continuous, but the converse does not hold.

3 Representation of Implications on $\mathscr{L}^I$

Similarly as for t-norms we can introduce a direct representability for implications on $\mathscr{L}^I$, as well as optimistic and pessimistic representability, by means of implications on $([0,1], \leq)$.

Lemma 2. *[21] Given implications I_1 and I_2 on $([0,1], \leq)$ with $I_1 \leq I_2$, the mapping $\mathscr{I}_{I_1,I_2} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*
$$\mathscr{I}_{I_1,I_2}(x,y) = [I_1(x_2, y_1), I_2(x_1, y_2)]$$
is an implication on $\mathscr{L}^I$.

Note that $\mathscr{I}_{I,I}$ is the best interval representation of I. Furthermore, for continuous I it holds that
$$\mathscr{I}_{I,I}([x_1,x_2],[y_1,y_2]) = \{I(\alpha,\beta) \mid \alpha \in [x_1,x_2] \text{ and } \beta \in [y_1,y_2]\}.$$

Lemma 3. *[21] Given an implication I on $([0,1],\leq)$ the mappings $\mathscr{I}_I : (L^I)^2 \to L^I$ and $\mathscr{I}'_I : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathscr{I}_I = [I(x_2,y_1), \max(I(x_1,y_1), I(x_2,y_2))],$$
$$\mathscr{I}'_I = [\min(I(x_1,y_1), I(x_2,y_2)), I(x_1,y_2)],$$

are implications on $\mathscr{L}^I$.

Definition 9. [21] Let I_1, I_2 and I be implications on $([0,1],\leq)$, the mappings $\mathscr{I}_{I_1,I_2}$, $\mathscr{I}_I$ and $\mathscr{I}'_I$ defined in Lemma 2 and 3 are called the i-representable implication on $\mathscr{L}^I$ with representatives I_1 and I_2, the pessimistic and the optimistic implication on $\mathscr{L}^I$ with representative I, respectively.

Implications on $\mathscr{L}^I$ can also be generated from t-(co)norms and negations as S- and R-implications. We study the relationship of these constructs to i-representability and optimistic and pessimistic representability.

The following proposition shows that there exists a strong relationship between S-implications on $\mathscr{L}^I$ based on a t-representable t-conorm and S-implications on the unit interval based on the representatives of that t-conorm.

Proposition 4. *[6] A mapping $\mathscr{I} : (L^I)^2 \to L^I$ is an S-implication based on an involutive negation $\mathscr{N}_{N,N}$ and on a t-representable t-conorm $\mathscr{S}_{S_1,S_2}$ if, and only if, there exist S-implications $I_{S_1,N}, I_{S_2,N} : [0,1]^2 \to [0,1]$ based on the negation N and the t-conorms S_1 and S_2 respectively, such that*

$$\mathscr{I}([x_1,x_2],[y_1,y_2]) = [I_{S_1,N}(x_2,y_1), I_{S_2,N}(x_1,y_2)].$$

So, S-implications on $\mathscr{L}^I$ generated by a t-representable t-conorm and an involutive negation are i-representable implications having an S-implication on $([0,1],\leq)$ as their representative. For R-implications, no such transparent relation with i-representability exists.

Proposition 5. *[21] No R-implication on $\mathscr{L}^I$ is i-representable.*

We discuss now how optimistic and pessimistic implications can be related to optimistic and pessimistic t-norms through the construction of the corresponding R- and S-implications.

Proposition 6. *[21] Let $\mathscr{T}_T$ be a pessimistic t-norm on $\mathscr{L}^I$. Then the R-implication generated by $\mathscr{T}_T$ is given by the optimistic implication with representative I_T, i.e.*

$$\mathscr{I}_{\mathscr{T}_T} = \mathscr{I}'_{I_T}.$$

Proposition 7. *[21] Let $\mathscr{T}'_T$ be an optimistic t-norm on $\mathscr{L}^I$. Then the R-implication generated by $\mathscr{T}'_T$ is given by, for all x, y in L^I,*

$$\mathscr{I}_{\mathscr{T}'_T}(x,y) = [\min(I_T(x_1,y_1), I_T(x_2,y_2)), I_T(x_2,y_2)].$$

This formula resembles the one corresponding to optimistic implications. However, the upper bound involves x_2 instead of x_1, so contrary to optimistic implicators this bound does not correspond to the highest possible value of $I(\alpha,\beta)$, where α,β in $[0,1]$. Obviously, $\mathscr{I}_{\mathscr{T}'_T}$ is not a pessimistic implication either. Moreover, it is equal to the R-implication generated by the corresponding t-representable t-norm $\mathscr{T}_{T,T}$. More generally, we have the following property.

Proposition 8. *[21] Let $\mathscr{T}_{T_1,T_2}$ be a t-representable t-norm on $\mathscr{L}^I$. Then the R-implication generated by $\mathscr{T}_{T_1,T_2}$ is given by, for all x,y in L^I,*

$$\mathscr{I}_{\mathscr{T}_{T_1,T_2}}(x,y) = [\min(I_{T_1}(x_1,y_1), I_{T_2}(x_2,y_2)), I_{T_2}(x_2,y_2)].$$

For the S-implications corresponding to pessimistic and optimistic t-conorms we obtain the following.

Proposition 9. *[21] Let $\mathscr{S}_S$ be a pessimistic t-conorm on $\mathscr{L}^I$ with representative S and let $\mathscr{N}_{N,N}$ be an n-representable negation with representative N. Then the S-implication generated by $\mathscr{S}_S$ and $\mathscr{N}_{N,N}$ is the pessimistic implication with representative $I_{S,N}$, i.e.*

$$\mathscr{I}_{\mathscr{S}_S,\mathscr{N}_{N,N}} = \mathscr{I}_{I_{S,N}}.$$

Let $\mathscr{S}'_S$ be an optimistic t-conorm on $\mathscr{L}^I$ with representative S and let $\mathscr{N}_{N,N}$ be an n-representable negation with representative N. Then the S-implication generated by $\mathscr{S}'_S$ and $\mathscr{N}_{N,N}$ is the optimistic implication with representative $I_{S,N}$, i.e.

$$\mathscr{I}_{\mathscr{S}'_S,\mathscr{N}_{N,N}} = \mathscr{I}'_{I_{S,N}}.$$

We see that pessimistic t-norms generate optimistic R-implications, but optimistic t-norms do not generate pessimistic implications. The R-implications generated by optimistic t-norms coincide with the R-implications generated by t-representable t-norms. However, no intuitive interpretation of these R-implications can be given. On the other hand, for S-implications the situation is clearer: pessimistic t-conorms generate pessimistic S-implications, optimistic t-conorms generate optimistic S-implications and t-representable t-conorms generate i-representable S-implications.

4 Smets-Magrez Axioms

In the previous section we have seen that the class of pessimistic t-norms is the only one which generate both R- and S-implications that belong to one of the classes of representable implications which we discussed before. The superiority of the pessimistic t-norms goes even further as we will see below.

Let $\mathscr{I}$ be an implication on $\mathscr{L}$. The mapping $\mathscr{N}_{\mathscr{I}} : L \to L$ defined by $\mathscr{N}_{\mathscr{I}}(x) = \mathscr{I}(x, 0_{\mathscr{L}})$, for all $x \in L$, is a negation on $\mathscr{L}$, called the negation generated by $\mathscr{I}$.

The Smets-Magrez axioms, a set of natural and commonly imposed criteria for implications on the unit interval, can be extended to $\mathscr{L}^I$ as follows [17]. An implication $\mathscr{I}$ on $\mathscr{L}^I$ is said to satisfy the Smets-Magrez axioms if for all x, y, z in L^I,

(A.1) $\mathscr{I}(., y)$ is decreasing and $\mathscr{I}(x, .)$ is increasing (monotonicity laws),
(A.2) $\mathscr{I}(1_{\mathscr{L}^I}, x) = x$ (neutrality principle),
(A.3) $\mathscr{I}(\mathscr{N}_{\mathscr{I}}(y), \mathscr{N}_{\mathscr{I}}(x)) = \mathscr{I}(x, y)$ (contrapositivity),
(A.4) $\mathscr{I}(x, \mathscr{I}(y, z)) = \mathscr{I}(y, \mathscr{I}(x, z))$ (exchange principle),
(A.5) $\mathscr{I}(x, y) = 1_{\mathscr{L}^I} \iff x \leq_{L^I} y$ (confinement principle),
(A.6) $\mathscr{I}$ is a continuous $(L^I)^2 \to L^I$ mapping (continuity).

Note that according to our definition, any implication on $\mathscr{L}^I$ satisfies (A.1).

Proposition 10. *[17] An S-implication $\mathscr{I}_{\mathscr{S},\mathscr{N}}$ on $\mathscr{L}^I$ satisfies (A.2), (A.3) and (A.4) if and only if $\mathscr{N}$ is involutive.*

Proposition 11. *[17] An S-implication $\mathscr{I}_{\mathscr{S},\mathscr{N}}$ on $\mathscr{L}^I$ satisfies (A.6) as soon as $\mathscr{S}$ and $\mathscr{N}$ are continuous.*

The following proposition shows that only studying i-representable implicators reduces the possibilities of finding an implication on $\mathscr{L}^I$ which satisfies all Smets-Magrez axioms.

Proposition 12. *[17] Axiom (A.5) fails for every S-implication $\mathscr{I}_{\mathscr{S},\mathscr{N}}$ on $\mathscr{L}^I$ for which $\mathscr{S}$ is t-representable and $\mathscr{N}$ is involutive.*

For R-implications on $\mathscr{L}^I$ we have the following results.

Proposition 13. *[17] Every R-implication $\mathscr{I}_{\mathscr{T}}$ on $\mathscr{L}^I$ satisfies (A.2).*

Proposition 14. *[17] An R-implication $\mathscr{I}_{\mathscr{T}}$ on $\mathscr{L}^I$ satisfies (A.5) if and only if there exists for each $x = [x_1, x_2] \in L^I$ a sequence $(\delta_i)_{i \in \mathbb{N} \setminus \{0\}}$ in $\Omega = \{\delta \mid \delta \in L^I$ and $\delta_2 < 1\}$ such that $\lim_{i \to +\infty} \delta_i = 1_{\mathscr{L}^I}$ and*

$$\lim_{i \to +\infty} \mathrm{pr}_1\, \mathscr{T}(x, \delta_i) = x_1,$$
$$\lim_{i \to +\infty} \mathrm{pr}_2\, \mathscr{T}(x, \delta_i) = x_2.$$

As a consequence of the last proposition, if $\mathscr{T}$ is a t-norm on $\mathscr{L}^I$ for which $\mathrm{pr}_1\, \mathscr{T} : (L^I)^2 \to [0,1]$ and $\mathrm{pr}_2\, \mathscr{T} : (L^I)^2 \to [0,1]$ are left-continuous mappings, then $\mathscr{I}_{\mathscr{T}}$ satisfies (A.5).

Similarly as for S-implications, limiting ourselves to R-implications generated by t-representable t-norms reduces our chances of finding an implication which satisfies all Smets-Magrez axioms.

Proposition 15. *[17] Axiom (A.3) fails for every R-implication $\mathscr{I}_{\mathscr{T}}$ on $\mathscr{L}^I$ for which $\mathscr{T}$ is t-representable.*

Similarly as for t-norms on the unit interval we have the following property.

Proposition 16. *[17] If an implication $\mathscr{I}$ on $\mathscr{L}^I$ satisfies (A.2), (A.3), (A.4), then the mappings $\mathscr{T}_{\mathscr{I}}, \mathscr{S}_{\mathscr{I}} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathscr{T}_{\mathscr{I}}(x,y) = \mathscr{N}_{\mathscr{I}}(\mathscr{I}(x, \mathscr{N}_{\mathscr{I}}(y))),$$
$$\mathscr{S}_{\mathscr{I}}(x,y) = \mathscr{I}(\mathscr{N}_{\mathscr{I}}(x), y),$$

are a t-norm and a t-conorm on L^I, respectively.

As a consequence, all implications on $\mathscr{L}^I$ satisfying (A.2), (A.3) and (A.4) are S-implications.

We check for the class of t-norms $\mathscr{T}_{T,t}$ under which conditions the residual implication $\mathscr{I}_{\mathscr{T}_{T,t}}$ satisfies the Smets-Magrez axioms.

Proposition 17. *[27] Let T be a t-norm on $([0,1], \leq)$ and $t \in [0,1]$. The residual implication $\mathscr{I}_{\mathscr{T}_{T,t}}$ of $\mathscr{T}_{T,t}$ satisfies*

- *(A.1) and (A.2);*
- *(A.3) if and only if $t = 1$ and I_T satisfies (A.3);*
- *(A.4) if and only if I_T satisfies (A.4);*
- *(A.5) if and only if I_T satisfies (A.5);*
- *(A.6) as soon as T is continuous and I_T satisfies (A.6).*

The main result of this section says that the implications on $\mathscr{L}^I$ which satisfy all Smets-Magrez axioms and the additional border condition $\mathscr{I}(D,D) \subseteq D$ (which means that all exact intervals are mapped on exact intervals, or, in other words, that an implication can not add uncertainty when there is no uncertainty in the original values) can be fully characterized in terms of the residual implication of the pessimistic extension of the Łukasiewicz t-norm.

Proposition 18. *[17] An implication $\mathscr{I}$ on $\mathscr{L}^I$ satisfies all Smets-Magrez axioms and $\mathscr{I}(D,D) \subseteq D$ if and only if there exists a continuous increasing permutation Φ of L^I with increasing inverse such that for all x, y in L^I,*

$$\mathscr{I}(x,y) = \Phi^{-1}(\mathscr{I}_{\mathscr{T}_{T_L}}(\Phi(x), \Phi(y))).$$

5 Distributivity of Implication Functions Over Triangular Norms and Conorms

In this section we discuss the solutions of equations of the following kind:

$$\mathscr{I}(x, g(y,z)) = g(\mathscr{I}(x,y), \mathscr{I}(x,z)),$$

where $\mathscr{I}$ is an implication function on $\mathscr{L}^I$ and g is a t-norm or a t-conorm on $\mathscr{L}^I$. Distributivity of implications on the unit interval over different fuzzy logic connectives has been studied in the recent past by many authors (see [5, 10, 36, 41, 42, 46]). This interest, perhaps, was kickstarted by Combs and Andrews [16] which exploit the classical tautology

$$(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r)$$

in their inference mechanism towards reduction in the complexity of fuzzy "If–Then" rules.

We say that a t-norm T on $([0,1], \leq)$ is strict, if it is continuous and strictly monotone, i.e. $T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$.

Proposition 19. *A function $T : [0,1]^2 \rightarrow [0,1]$ is a strict t-norm if and only if there exists a continuous, strictly decreasing function $t : [0,1] \rightarrow [0,+\infty]$ with $t(1) = 0$ and $t(0) = +\infty$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x,y) = t^{-1}(t(x) + t(y)), \quad \text{for all } (x,y) \in [0,1]^2.$$

The function t is called an additive generator of T.

In order to be able to find the implications on $\mathscr{L}^I$ which are distributive w.r.t. a t-representable t-norm generated from strict t-norms, we consider the following lemma.

Lemma 4. *[8] Let $L^\infty = \{(u_1,u_2) \mid (u_1,u_2) \in [0,+\infty]^2$ and $u_1 \geq u_2\}$. For a function $f : L^\infty \rightarrow [0,+\infty]$ the following statements are equivalent:*

1. *f satisfies the functional equation*

$$f(u_1+v_1, u_2+v_2) = f(u_1,u_2) + f(v_1,v_2), \quad \text{for all } (u_1,u_2), (v_1,v_2) \text{ in } L^\infty;$$

2. *either $f = 0$, or $f = +\infty$, or*

$$f(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0, \\ +\infty, & \text{else,} \end{cases}$$

or

$$f(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 < +\infty, \\ +\infty, & \text{else,} \end{cases}$$

or

$$f(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = 0, \\ +\infty, & \text{else,} \end{cases}$$

or

$$f(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 = u_2 < +\infty, \\ +\infty, & \text{else,} \end{cases}$$

or

$$f(u_1,u_2) = \begin{cases} 0, & \text{if } u_2 = 0 \text{ and } u_1 < +\infty, \\ +\infty, & \text{else,} \end{cases}$$

or

$$f(u_1,u_2) = \begin{cases} 0, & \text{if } u_1 < +\infty, \\ +\infty, & \text{else}, \end{cases}$$

or there exists a unique $c \in]0,+\infty[$ *such that* $f(u_1,u_2) = cu_1$, *or* $f(u_1,u_2) = cu_2$, *or*

$$f(u_1,u_2) = \begin{cases} cu_1, & \text{if } u_1 = u_2, \\ +\infty, & \text{else}, \end{cases}$$

or

$$f(u_1,u_2) = \begin{cases} cu_2, & \text{if } u_1 < +\infty, \\ +\infty, & \text{else}, \end{cases}$$

or

$$f(u_1,u_2) = \begin{cases} cu_1, & \text{if } u_2 = 0, \\ +\infty, & \text{else}, \end{cases}$$

or

$$f(u_1,u_2) = \begin{cases} c(u_1 - u_2), & \text{if } u_2 < +\infty, \\ +\infty, & \text{else}, \end{cases}$$

or there exist unique c_1, c_2 *in* $]0,+\infty[$ *with* $c_1 \neq c_2$ *such that*

$$f(u_1,u_2) = \begin{cases} c_1(u_1 - u_2) + c_2 u_2, & \text{if } u_2 < +\infty, \\ +\infty, & \text{else}, \end{cases}$$

for all $(u_1,u_2) \in L^\infty$.

The following proposition detailing the solutions of the distributivity equation follows immediately from the results in [8].

Proposition 20. *Let* $\mathscr{T}$ *be a t-representable t-norm generated from strict t-norms with generator* t_1 *and* t_2 *respectively. If a function* $\mathscr{I} : (L^I)^2 \to L^I$ *satisfies the equation*

$$\mathscr{I}(x, \mathscr{T}(y,z)) = \mathscr{T}(\mathscr{I}(x,y), \mathscr{I}(x,z)), \quad \text{for all } (x,y,z) \in (L^I)^3, \tag{5}$$

then for each $[x_1,x_2] \in L^I$*, each of the functions defined by, for all* $(a,b) \in L^\infty$,

$$f_{[x_1,x_2]}(a,b) = t_1 \circ \mathrm{pr}_1 \circ \mathscr{I}([x_1,x_2],[t_1^{-1}(a),t_1^{-1}(b)]),$$
$$f^{[x_1,x_2]}(a,b) = t_2 \circ \mathrm{pr}_2 \circ \mathscr{I}([x_1,x_2],[t_2^{-1}(a),t_2^{-1}(b)])$$

satisfies one of the representations given in Lemma 4.

In [7] the functions $\mathscr{I} : (L^I)^2 \to L^I$ which are continuous w.r.t. the second argument and which satisfy (5) are listed in the case that $\mathscr{T}$ is the t-representable t-norm generated from the product t-norm T_P on $([0,1],\leq)$.

Not all possibilities for f in Lemma 4 yield a mapping $\mathscr{I}$ which returns values in $\mathscr{L}^I$; furthermore the mappings $\mathscr{I}$ that do only return values in $\mathscr{L}^I$ are not all implications on $\mathscr{L}^I$ [8]. The following example shows that there is at least one possibility for f which produces an implication on $\mathscr{L}^I$.

Example 5. Let $[x_1,x_2]$ be arbitrary in L^I. Define for all $(a,b) \in L^\infty$,

$$f_{[x_1,x_2]}(a,b) = x_2 a,$$
$$f^{[x_1,x_2]}(a,b) = x_1 b.$$

We find

$$\mathscr{I}(x,y) = \begin{cases} 1_{\mathscr{L}^I}, & \text{if } x_2 = 0, \\ [t_1^{-1}(x_2 t_1(y_1)), 1], & \text{if } x_1 = 0 < x_2, \\ [t_1^{-1}(x_2 t_1(y_1)), t_2^{-1}(x_1 t_2(y_2))], & \text{otherwise.} \end{cases}$$

Note that $\mathscr{I} = \mathscr{I}_{I_1,I_2}$ where for $i \in \{1,2\}$, I_i is the implication on $([0,1],\leq)$ defined by, for all x,y in $[0,1]$,

$$I_i(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ t_i^{-1}(x t_i(y)), & \text{otherwise.} \end{cases}$$

It can be straightforwardly verified that the implication $\mathscr{I} = \mathscr{I}_{I_1,I_2}$ and the t-representable t-norm generated by strict t-norms with generators t_1 and t_2 satisfy (5). For $T_1 = T_2 = T_P$ we have that $t_1(x) = t_2(x) = -\ln(x)$ (so $t_1^{-1}(a) = t_2^{-1}(a) = e^{-a}$) and we obtain

$$\mathscr{I}(x,y) = \begin{cases} 1_{\mathscr{L}^I}, & \text{if } x_2 = y_1 = 0, \\ [y_1^{x_2}, 1], & \text{if } x_1 = y_2 = 0 < x_2, \\ [y_1^{x_2}, y_2^{x_1}], & \text{otherwise,} \end{cases}$$

This implication resembles the function $\mathscr{I}$ in Example 11 of [7]; however the latter is not increasing in its second argument and therefore not an implication on $\mathscr{L}^I$. Indeed, $\mathscr{I}([0,1],[0,0]) = 1_{\mathscr{L}^I}$ and $\mathscr{I}([0,1],[y_1,y_1]) = [y_1,1] \ngeq_{L^I} \mathscr{I}([0,1],[0,0])$ for any $y_1 \in]0,1[$.

6 Interval-Valued Residuated Lattices, Triangle Algebras and Interval-Valued Monoidal Logic

In this section we relate implications on $\mathscr{L}^I$ to a generalization of fuzzy logic to the interval-valued fuzzy case. We first discuss triangle algebras which are special cases of residuated lattices designed for being used in interval-valued fuzzy set theory.

6.1 Interval-Valued Residuated Lattices and Triangle Algebras

We consider special cases of residuated lattices in which new operators are added so that the resulting structure captures the triangular shape of $\mathscr{L}^I$ (and its generalizations). First we recall the definition of a residuated lattice.

Definition 10. [28] A residuated lattice is a structure $\mathscr{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ in which $\sqcap$, $\sqcup$, $*$ and $\Rightarrow$ are binary operators on the set L and

- $(L, \sqcap, \sqcup, 0, 1)$ is a bounded lattice (with 0 as smallest and 1 as greatest element),
- $*$ is commutative and associative, with 1 as neutral element, and
- $x * y \leq z$ iff $x \leq y \Rightarrow z$ for all x, y and z in L (residuation principle).

The binary operations $*$ and $\Rightarrow$ are called product and implication, respectively. We will use the notations $\neg x$ for $x \Rightarrow 0$ (negation), $x \iff y$ for $(x \Rightarrow y) \sqcap (y \Rightarrow x)$ and x^n for $\underbrace{x * x * \cdots * x}_{n \text{ times}}$.

Definition 11. [49] Given a lattice $\mathscr{L} = (L, \sqcap, \sqcup)$ (called the base lattice), its triangularization $\mathbb{T}(\mathscr{L})$ is the structure $\mathbb{T}(\mathscr{L}) = (\mathrm{Int}(\mathscr{L}), \bigsqcap, \bigsqcup)$ defined by

- $\mathrm{Int}(\mathscr{L}) = \{[x_1, x_2] \mid (x_1, x_2) \in L^2 \text{ and } x_1 \leq x_2\}$,
- $[x_1, x_2] \bigsqcap [y_1, y_2] = [x_1 \sqcap y_1, x_2 \sqcap y_2]$,
- $[x_1, x_2] \bigsqcup [y_1, y_2] = [x_1 \sqcup y_1, x_2 \sqcup y_2]$.

The set $D_{\mathscr{L}} = \{[x, x] \mid x \in L\}$ is called the set of exact elements of $\mathbb{T}(\mathscr{L})$.

Definition 12. [49] An interval-valued residuated lattice (IVRL) is a residuated lattice $(\mathrm{Int}(\mathscr{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ on the triangularization $\mathbb{T}(\mathscr{L})$ of a bounded lattice $\mathscr{L} = (L, \cap, \cup)$ in which $D_{\mathscr{L}}$ is closed under $\odot$ and $\Rightarrow_\odot$, i.e. $[x,x] \odot [y,y] \in D_{\mathscr{L}}$ and $[x,x] \Rightarrow_\odot [y,y] \in D_{\mathscr{L}}$ for all x, y in L.

When we add $[0,1]$ as a constant, and p_v and p_h (defined by $p_v([x_1, x_2]) = [x_1, x_1]$ and $p_h([x_1, x_2]) = [x_2, x_2]$ for all $[x_1, x_2]$ in $\mathrm{Int}(\mathscr{L})$) as unary operators, the structure $(\mathrm{Int}(\mathscr{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, p_v, p_h, [0,0], [0,1], [1,1])$ is called an extended IVRL.

Example 6. Let T be a left-continuous t-norm on $([0,1], \min, \max)$, $t \in [0,1]$. Then $(L^I, \inf, \sup, \mathscr{T}_{T,t}, \mathscr{I}_{\mathscr{T}_{T,t}}, [0,0], [1,1])$ is an IVRL.

The triangular norms $\mathscr{T}$ on $\mathscr{L}^I$ satisfying the residuation principle and which satisfy the property that D is closed under $\mathscr{T}$ and $\mathscr{I}_{\mathscr{T}}$ are completely characterized in terms of a t-norm T on the unit interval.

Proposition 21. *[49] Let* $(\mathrm{Int}(\mathscr{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ *be an IVRL and let* $t \in L$, $* : L^2 \to L$ *and* $\Rightarrow : L^2 \to L$ *be defined by*

$$\begin{aligned} t &= \mathrm{pr}_2([0,1] \odot [0,1]), \\ x * y &= \mathrm{pr}_1([x,x] \odot [y,y]), \\ x \Rightarrow y &= \mathrm{pr}_1([x,x] \Rightarrow_\odot [y,y]), \end{aligned}$$

for all x, y in L. Then for all x, y in $\mathrm{Int}(\mathscr{L})$*,*

$$[x_1,x_2] \odot [y_1,y_2] = [x_1 * y_1, (x_2 * y_2 * t) \cup (x_1 * y_2) \cup (x_2 * y_1)],$$
$$[x_1,x_2] \Rightarrow_\odot [y_1,y_2] = [(x_1 \Rightarrow y_1) \cap (x_2 \Rightarrow y_2), (x_1 \Rightarrow y_2) \cap (x_2 \Rightarrow (t \Rightarrow y_2))].$$

To capture the triangular structure of IVRLs, we extend the definition of a residuated lattice with a new constant u ("uncertainty") and two new unary connectives ν ("necessity") and μ ("possibility"). Intuitively, the elements of a triangle algebra may be thought of as closed intervals, u as the interval $[0,1]$, and ν and μ as operators mapping $[x_1,x_2]$ to $[x_1,x_1]$ and $[x_2,x_2]$ respectively.

Definition 13. [47, 50] A triangular algebra is a structure $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ of type $(2,2,2,2,1,1,0,0,0)$ such that $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice and, for all x, y in A,

(T.1) $\nu x \leq x$,
(T.2) $\nu x \leq \nu\nu x$,
(T.3) $\nu(x \sqcap y) = \nu x \sqcap \nu y$,
(T.4) $\nu(x \sqcup y) = \nu x \sqcup \nu y$,
(T.5) $\nu u = 0$,
(T.6) $\nu\mu x = \mu x$,
(T.7) $\nu(x \Rightarrow y) \leq \nu x \Rightarrow \nu y$,
(T.8) $(\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leq (x \Leftrightarrow y)$,
(T.9) $\nu x \Rightarrow \nu y \leq \nu(\nu x \Rightarrow \nu y)$,

(T.1') $x \leq \mu x$,
(T.2') $\mu\mu x \leq \mu x$,
(T.3') $\mu(x \sqcap y) = \mu x \sqcap \mu y$,
(T.4') $\mu(x \sqcup y) = \mu x \sqcup \mu y$,
(T.5') $\mu u = 1$,
(T.6') $\mu\nu x = \nu x$,

where the biresiduum $\Leftrightarrow$ is defined as $x \Leftrightarrow y = x \Rightarrow y \wedge y \Rightarrow x$, for all x, y in A. The unary operators ν and μ are called the necessity and possibility operator, respectively.

Note that in a triangle algebra $x = \nu x \sqcup (\mu x \sqcap u)$, for all $x \in A$. This shows that an element of the triangle algebra is completely determined by its necessity and its possibility.

There is a one-to-one correspondance between the class of IVRLs and the class of triangle algebras.

Proposition 22. *[50] Every triangle algebra is isomorphic to an extended IVRL (see Figure 2). Conversely, every extended IVRL is a triangular algebra.*

6.2 *Interval-Valued Monoidal Logic*

We now translate the defining properties of triangle algebras into logical axioms and show that the resulting logic IVML is sound and complete w.r.t. the variety of triangle algebras.

The language of IVML consists of countably many proposition variables (p_1, p_2, ...), the constants $\overline{0}$ and $\overline{u}$, the unary operators $\Box$, $\Diamond$, the binary operators $\wedge$, $\vee$, &, $\rightarrow$, and finally the auxiliary symbols '(' and ')'. Formulas are defined inductively:

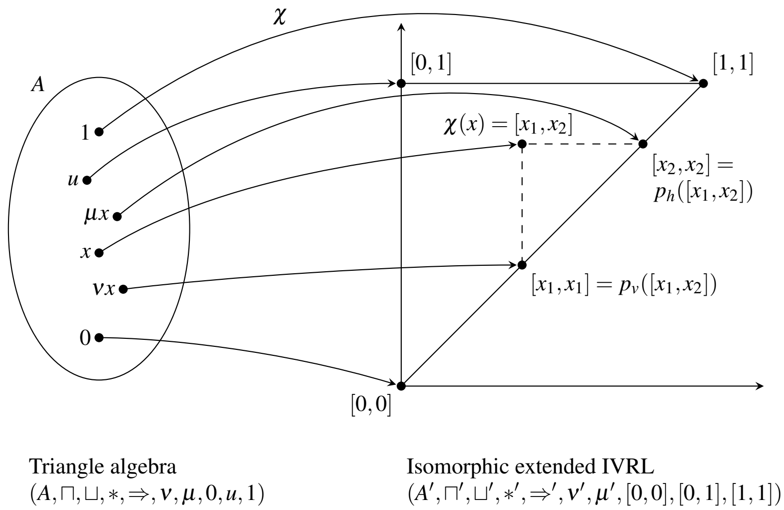

Fig. 2 The isomorphism ξ from a triangle algebra to an IVRL.

proposition variables, $\overline{0}$ and $\overline{u}$ are formulas; if ϕ and ψ are formulas, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \& \psi)$, $(\phi \rightarrow \psi)$, $(\Box \psi)$ and $(\Diamond \psi)$.

In order to avoid unnecessary brackets, we agree on the following priority rules:

- unary operators always take precedence over binary ones, while
- among the binary operators, & has the highest priority; furthermore $\wedge$ and $\vee$ take precedence over $\rightarrow$,
- the outermost brackets are not written.

We also introduce some useful shorthand notations: $\overline{1}$ for $\overline{0} \rightarrow \overline{0}$, $\neg\phi$ for $\phi \rightarrow \overline{0}$ and $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ for formulas ϕ and ψ.

The axioms of IVML are those of ML (Monoidal Logic) [35], i.e.

$$
\begin{array}{ll}
(ML.1) & (\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi)), \\
(ML.2) & \phi \rightarrow (\phi \vee \psi), \\
(ML.3) & \psi \rightarrow (\phi \vee \psi), \\
(ML.4) & (\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\phi \vee \psi) \rightarrow \chi)), \\
(ML.5) & (\phi \wedge \psi) \rightarrow \phi, \\
(ML.6) & (\phi \wedge \psi) \rightarrow \psi, \\
(ML.7) & (\phi \& \psi) \rightarrow \phi, \\
(ML.8) & (\phi \& \psi) \rightarrow (\psi \& \phi), \\
(ML.9) & (\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \wedge \chi))), \\
(ML.10) & (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \& \psi) \rightarrow \chi), \\
(ML.11) & ((\phi \& \psi) \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi)), \\
(ML.12) & \overline{0} \rightarrow \phi,
\end{array}
$$

complemented with

(*IVML*.1) $\Box\phi \to \phi$, (*IVML*.1′) $\phi \to \Diamond\phi$,
(*IVML*.2) $\Box\phi \to \Box\Box\phi$, (*IVML*.2′) $\Diamond\Diamond\phi \to \Diamond\phi$,
(*IVML*.3) $(\Box\phi \wedge \Box\psi) \to \Box(\phi \wedge \psi)$, (*IVML*.3′) $(\Diamond\phi \wedge \Diamond\psi) \to \Diamond(\phi \wedge \psi)$,
(*IVML*.4) $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$, (*IVML*.4′) $\Diamond(\phi \vee \psi) \to (\Diamond\phi \vee \Diamond\psi)$,
(*IVML*.5) $\Box\overline{1}$, (*IVML*.5′) $\neg\Diamond\overline{0}$,
(*IVML*.6) $\neg\Box\overline{u}$, (*IVML*.6′) $\Diamond\overline{u}$,
(*IVML*.7) $\Diamond\phi \to \Box\Diamond\phi$, (*IVML*.7′) $\Diamond\Box\phi \to \Box\phi$,
(*IVML*.8) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$,
(*IVML*.9) $(\Box\phi \leftrightarrow \Box\psi)\&(\Diamond\phi \leftrightarrow \Diamond\psi) \to (\phi \leftrightarrow \psi)$,
(*IVML*.10) $(\Box\phi \to \Box\psi) \to \Box(\Box\phi \to \Box\psi)$.

The deduction rules are modus ponens (MP, from ϕ and $\phi \to \psi$ infer ψ), generalization[2] (G, from ϕ infer $\Box\phi$) and monotonicity of $\Diamond$ (M$\Diamond$, from $\phi \to \psi$ infer $\Diamond\phi \to \Diamond\psi$).

The consequence relation $\vdash$ is defined as follows, in the usual way. Let V be a theory, i.e., a set of formulas in IVML. A (formal) proof of a formula ϕ in V is a finite sequence of formulas with ϕ at its end, such that every formula in the sequence is either an axiom of IVML, a formula of V, or the result of an application of an inference rule to previous formulas in the sequence. If a proof for ϕ exists in V, we say that ϕ can be deduced from V and we denote this by $V \vdash \phi$.
For a theory V, and formulas ϕ and ψ in IVML, denote $\phi \sim_V \psi$ iff $V \vdash \phi \to \psi$ and $V \vdash \psi \to \phi$ (this is also equivalent with $V \vdash \phi \leftrightarrow \psi$).

Note that (IVML.5) is in fact superfluous, as it immediately follows from $\emptyset \vdash \overline{1}$ and generalization; we include it here to obtain full correspondence with Definition 13.

Definition 14. Let $\mathscr{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and V a theory. An $\mathscr{A}$-evaluation is a mapping e from the set of formulas of IVML to A that satisfies, for each two formulas ϕ and ψ: $e(\phi \wedge \psi) = e(\phi) \sqcap e(\psi)$, $e(\phi \vee \psi) = e(\phi) \sqcup e(\psi)$, $e(\phi \& \psi) = e(\phi) * e(\psi)$, $e(\phi \to \psi) = e(\phi) \Rightarrow e(\psi)$, $e(\Box\phi) = \nu e(\phi)$, $e(\Diamond\phi) = \mu e(\phi)$, $e(\overline{0}) = 0$ and $e(\overline{u}) = u$. If an $\mathscr{A}$-evaluation e satisfies $e(\chi) = 1$ for every χ in V, it is called an $\mathscr{A}$-model for V.

The following property shows that interval-valued monoidal logic is sound w.r.t. the variety of triangle algebras, i.e., that if a formula ϕ can be deduced from a theory V in IVML, then for every triangle algebra $\mathscr{A}$ and for every $\mathscr{A}$-model e of V, $e(\phi) = 1$, and that IVML is also complete (i.e. that the converse of soundness also holds).

Proposition 23 (Soundness and completeness of IVML). *[50] A formula ϕ can be deduced from a theory V in IVML iff for every triangle algebra $\mathscr{A}$ and for every $\mathscr{A}$-model e of V, $e(\phi) = 1$.*

By adding axioms, we can obtain axiomatic extensions of interval-valued monoidal logic. For these extensions a similar soundness and completeness result holds.

[2] Generalization is often called necessitation, e.g. in [59]

Furthermore, in some cases we can obtain a stronger result. For example, similarly as for monoidal t-norm based logic (MTL) [32] we obtain the following result.

Proposition 24 (Standard completeness). *[51] For each formula ϕ, the following three statements are equivalent:*

- *ϕ can be deduced from a theory V in IVMTL (which is the axiomatic extension of IVML obtained by adding the axiom scheme $(\Box\phi \to \Box\psi) \vee (\Box\psi \to \Box\phi)$),*
- *for every triangle algebra $\mathscr{A}$ in which the set of exact elements is prelinear and for every $\mathscr{A}$-model e of V, $e(\phi) = 1$,*
- *for every triangle algebra $\mathscr{A}$ in which the set of exact elements is linear and for every $\mathscr{A}$-model e of V, $e(\phi) = 1$.*

More information on this and on the soundness and completeness of other axiomatic extensions of IVML can be found in [47, 51, 52].

Remark 1. Interval-valued monoidal logic is a truth-functional logic: the truth degree of a compound proposition is determined by the truth degree of its parts. This causes some counterintuitive results, if we want to interpret the element $[0,1]$ of an IVRL as uncertainty. For example: suppose we don't know anything about the truth value of propositions p and q, i.e., $v(p) = v(q) = [0,1]$. Then yet the implication $p \to q$ is definitely valid: $v(p \to q) = v(p) \Rightarrow v(q) = [1,1]$. However, if $\neg[0,1] = [0,1]$ [3] (which is intuitively preferable, since the negation of an uncertain proposition is still uncertain), then we can take $q = \neg p$, and obtain that $p \to \neg p$ is true. Or, equivalently (using the residuation principle), that $p\&p$ is false. This does not seem intuitive, as one would rather expect $p\&p$ to be uncertain if p is uncertain. Another consequence of $[0,1] \Rightarrow [0,1] = [1,1]$ is that it is impossible to interpret the intervals as a set in which the 'real' (unknown) truth value is contained, and $X \Rightarrow Y$ as the smallest closed interval containing every $x \Rightarrow y$, with x in X and y in Y (as in [31]). Indeed: $1 \in [0,1]$ and $0 \in [0,1]$, but $1 \Rightarrow 0 = 0 \notin [1,1]$.
On the other hand, for t-norms it is possible that $X * Y$ is the smallest closed interval containing every $x * y$, with x in X and y in Y, but only if they are t-representable (described by the axiom $\mu(x * y) = \mu x * \mu y$). However, in this case $\neg[0,1] = [0,0]$, which does not seem intuitive ('the negation of an uncertain proposition is absolutely false').
These considerations seem to suggest that interval-valued monoidal logic is not suitable to reason with uncertainty. This does not mean that intervals are not a good way for representing degrees of uncertainty, only that they are not suitable as truth values in a truth functional logical calculus when we interpret them as expressing uncertainty. It might even be impossible to model uncertainty as a truth value in any truth-functional logic. This question is discussed in [29, 30]. However, nothing prevents the intervals in interval-valued monoidal logic from having more adequate interpretations.

[3] This is for example the case if $\neg$ is involutive.

7 Other Constructions of Implications on $\mathscr{L}^I$

In the previous section we have seen that implications on $\mathscr{L}^I$ can be constructed as R- or S-implications derived from t-norms or t-conorms on $\mathscr{L}^I$. In this section we describe several other constructions of implications on $\mathscr{L}^I$.

7.1 Conjugacy

In Proposition 3 we have seen that an increasing permutation on $\mathscr{L}^I$ which has an increasing inverse is completely determined by a permutation on $([0,1],\leq)$. Such permutations can be used to construct new functions as follows.

We say that the functions $F,G:(L^I)^2 \to L^I$ are conjugate, if there exists an increasing bijection $\Phi : L^I \to L^I$ with increasing inverse such that $G = F_\Phi$, where

$$F_\phi(x,y) = \Phi^{-1}(F(\Phi(x),\Phi(y))), \quad \text{for all } (x,y) \in (L^I)^2.$$

Proposition 25. *[6] Let $\Phi : L^I \to L^I$ be an increasing permutation with increasing inverse.*

- *If $\mathscr{I}$ is an implication on $\mathscr{L}^I$, then $\mathscr{I}_\Phi$ is an implication on $\mathscr{L}^I$.*
- *If $\mathscr{I}$ is an S-implication on $\mathscr{L}^I$ based on some t-conorm $\mathscr{S}$ and strong negation $\mathscr{N}$ on $\mathscr{L}^I$, then $\mathscr{I}_\Phi$ is also an S-implication on $\mathscr{L}^I$ based on the t-conorm $\mathscr{S}_\Phi$ and strong negation $\mathscr{N}_\Phi$.*
- *If $\mathscr{I}$ is an R-implication on $\mathscr{L}^I$ based on some t-norm $\mathscr{T}$ on $\mathscr{L}^I$, then $\mathscr{I}_\Phi$ is also an R-implication on $\mathscr{L}^I$ based on the t-norm $\mathscr{T}_\Phi$.*

7.2 Implications Defined Using Arithmetic Operators on $\mathscr{L}^I$

Let $\bar{L}^I = \{[x_1,x_2] \mid (x_1,x_2) \in \mathbb{R}^2 \text{ and } x_1 \leq x_2\}$ and $\bar{L}^I_+ = \{[x_1,x_2] \mid (x_1,x_2) \in [0,+\infty[^2$ and $x_1 \leq x_2\}$. We start from two arithmetic operators $\oplus : (\bar{L}^I)^2 \to \bar{L}^I$ and $\otimes : (\bar{L}^I_+)^2 \to \bar{L}^I$ satisfying the following properties (see [20]),

(ADD-1) $\oplus$ is commutative,
(ADD-2) $\oplus$ is associative,
(ADD-3) $\oplus$ is increasing,
(ADD-4) $0_{\mathscr{L}^I} \oplus a = a$, for all $a \in \bar{L}^I$,
(ADD-5) $[\alpha,\alpha] \oplus [\beta,\beta] = [\alpha+\beta,\alpha+\beta]$, for all α,β in $\mathbb{R}$,
(MUL-1) $\otimes$ is commutative,
(MUL-2) $\otimes$ is associative,
(MUL-3) $\otimes$ is increasing,
(MUL-4) $1_{\mathscr{L}^I} \otimes a = a$, for all $a \in \bar{L}^I_+$,
(MUL-5) $[\alpha,\alpha] \otimes [\beta,\beta] = [\alpha\beta,\alpha\beta]$, for all α,β in $[0,+\infty[$.

The conditions (ADD-1)–(ADD-4) and (MUL-1)–(MUL-4) are natural conditions for any addition and multiplication operators. The conditions (ADD-5) and (MUL-5) ensure that these operators are natural extensions of the addition and multiplication of real numbers to $\bar{L}^I$.

The mapping $\ominus$ is defined in [20] by, for all x, y in $\bar{L}^I$,

$$1_{\mathscr{L}^I} \ominus x = [1 - x_2, 1 - x_1], \tag{6}$$
$$x \ominus y = 1_{\mathscr{L}^I} \ominus ((1_{\mathscr{L}^I} \ominus x) \oplus y). \tag{7}$$

Similarly, the mapping $\oslash$ is defined by, for all x, y in $\bar{L}^I_{+,0}$,

$$1_{\mathscr{L}^I} \oslash x = \left[\frac{1}{x_2}, \frac{1}{x_1}\right], \tag{8}$$
$$x \oslash y = 1_{\mathscr{L}^I} \oslash ((1_{\mathscr{L}^I} \oslash x) \otimes y). \tag{9}$$

The properties of these operators are studied in [20].

Using the arithmetic operators on $\mathscr{L}^I$ we can construct t-norms, t-conorms and implications on $\mathscr{L}^I$ which are generalizations of the Łukasiewicz t-norm, t-conorm and implication on the unit interval and which have a similar arithmetic expression as those functions.

Proposition 26. *[20] The mapping $\mathscr{S}_\oplus : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathscr{S}_\oplus(x, y) = \inf(1_{\mathscr{L}^I}, x \oplus y), \tag{10}$$

is a t-conorm on $\mathscr{L}^I$ if and only if $\oplus$ satisfies the following condition:

$$\begin{aligned} &(\forall (x,y,z) \in (L^I)^3) \\ &\Big(\Big((\inf(1_{\mathscr{L}^I}, x \oplus y) \oplus z)_1 < 1 \text{ and } (x \oplus y)_2 > 1\Big) \\ &\implies (\inf(1_{\mathscr{L}^I}, x \oplus y) \oplus z)_1 = (x \oplus \inf(1_{\mathscr{L}^I}, y \oplus z))_1\Big). \end{aligned} \tag{11}$$

Furthermore $\mathscr{S}_\oplus$ is a natural extension of S_L to L^I.

Proposition 27. *[20] The mapping $\mathscr{T}_\oplus : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathscr{T}_\oplus(x, y) = \sup(0_{\mathscr{L}^I}, x \ominus (1_{\mathscr{L}^I} \ominus y)), \tag{12}$$

is a t-norm on $\mathscr{L}^I$ if and only if $\oplus$ satisfies (11). Furthermore, $\mathscr{T}_\oplus$ is a natural extension of T_L to L^I.

The following theorem gives a simpler sufficient condition so that $\mathscr{S}_\oplus$ is a t-conorm and $\mathscr{T}_\oplus$ is a t-norm on $\mathscr{L}^I$.

Proposition 28. *[20] Assume that $\oplus$ satisfies the following condition:*

$$(\forall(x,y)\in \bar{L}^I_+ \times L^I)$$
$$\Big(\Big(([x_1,1]\oplus y)_1 < 1 \textit{ and } x_2 \in\,]1,2]\Big) \implies ([x_1,1]\oplus y)_1 = (x\oplus y)_1\Big). \tag{13}$$

Then the mappings $\mathscr{T}_\oplus, \mathscr{S}_\oplus : (L^I)^2 \to L^I$ defined by, for all x,y in L^I,

$$\mathscr{T}_\oplus(x,y) = \sup(0_{\mathscr{L}^I}, x\ominus(1_{\mathscr{L}^I}\ominus y)),$$
$$\mathscr{S}_\oplus(x,y) = \inf(1_{\mathscr{L}^I}, x\oplus y),$$

are a t-norm and a t-conorm on $\mathscr{L}^I$ respectively. Furthermore $\mathscr{T}_\oplus$ is a natural extension of T_L to L^I, and $\mathscr{S}_\oplus$ is a natural extension of S_L to L^I.

Note that $\mathscr{N}_s(x) = 1_{\mathscr{L}^I}\ominus x$, for all $x\in L^I$. So we obtain the following result.

Proposition 29. *Under the same conditions as in Proposition 26 or 28, the mapping $\mathscr{I}_{\mathscr{S}_\oplus,\mathscr{N}_s}$ defined by, for all x,y in L^I,*

$$\mathscr{I}_{\mathscr{S}_\oplus,\mathscr{N}_s}(x,y) = \mathscr{S}_\oplus(\mathscr{N}_s(x),y) = \inf(1_{\mathscr{L}^I}, (1_{\mathscr{L}^I}\ominus x)\oplus y),$$

is an implication on $\mathscr{L}^I$. Furthermore, $\mathscr{I}_{\mathscr{S}_\oplus,\mathscr{N}_s}$ is a natural extension of $\mathscr{I}_{S_L,N_s}$ to L^I.

7.3 Implications Generated by Binary Aggregation Functions

By modifying the definition of $\mathscr{T}_{T,T}$, $\mathscr{T}_T$, ... we can obtain new binary aggregation functions on $\mathscr{L}^I$ which are not t-norms or t-conorms. For example, let T and T' be t-norms and S and S' be t-conorms on $([0,1],\le)$ with $T\le T'$ and $S\le S'$, then we define for all x,y in L^I (see [26]),

$$\mathscr{A}_T(x,y) = [\min(T(x_1,y_2),T(y_1,x_2)), \max(T(x_1,y_2),T(y_1,x_2))],$$
$$\mathscr{A}_S(x,y) = [\min(S(x_1,y_2),S(y_1,x_2)), \max(S(x_1,y_2),S(y_1,x_2))],$$
$$\mathscr{A}'_{T,T'}(x,y) = [\min(T(x_1,y_2),T(y_1,x_2)), T'(x_2,y_2)],$$
$$\mathscr{A}'_{S,S'}(x,y) = [S(x_1,y_1), \max(S'(x_2,y_1),S'(y_2,x_1))].$$

Proposition 30. *[26, 37] Let T and T' be left-continuous t-norms with $T\le T'$, S and S' be t-conorms with $S\le S'$, and N be an involutive negation on $([0,1],\le)$. Then*

- *the residuum of $\mathscr{A}_T$ is equal to the residual implication of $\mathscr{T}_T$, i.e. $\mathscr{I}_{\mathscr{A}_T} = \mathscr{I}_{\mathscr{T}_T}$;*
- *the mapping $\mathscr{I}_{\mathscr{A}_S,\mathscr{N}} : (L^I)^2 \to L^I$ given by, for all x,y in L^I,*

  $$\mathscr{I}_{\mathscr{A}_S,\mathscr{N}_{N,N}}(x,y) = [\min(I_{S,N}(x_1,y_1), I_{S,N}(x_2,y_2)), \max(I_{S,N}(x_1,y_1), I_{S,N}(x_2,y_2))],$$

  is an implication on $\mathscr{L}^I$;

- *the residuum of $\mathscr{A}'_{T,T'}$ given by, for all x, y in L^I,*

$$\mathscr{I}_{\mathscr{A}'_{T,S}}(x,y) = [\min(I_T(x_1,y_1), I_{T'}(x_2,y_2)), I_{T'}(x_2,y_2)],$$

is an implication on $\mathscr{L}^I$;
- *the mapping $\mathscr{I}_{\mathscr{A}'_{S,S'},\mathscr{N}_{N,N}} : (L^I)^2 \to L^I$ given by, for all x, y in L^I,*

$$\mathscr{I}_{\mathscr{A}'_{S,S'},\mathscr{N}_{N,N}}(x,y) = [S(N(x_2),y_1), \max(S'(N(x_1),y_1), S'(N(x_2),y_2))],$$

is an implication on $\mathscr{L}^I$.

7.4 Implications Generated by Uninorms on $\mathscr{L}^I$

Similarly as for t-norms, implications can also be derived from uninorms. Uninorms are a generalization of t-norms and t-conorms for which the neutral element can be any element of $\mathscr{L}^I$.

Definition 15. [25] A uninorm on a complete lattice $\mathscr{L} = (L, \leq_L)$ is a commutative, associative, increasing mapping $\mathscr{U} : L^2 \to L$ which satisfies

$$(\exists e \in L)(\forall x \in L)(\mathscr{U}(e,x) = x).$$

The element e corresponding to a uninorm $\mathscr{U}$ is unique and is called the neutral element of $\mathscr{U}$.

If $\mathscr{U}(0_{\mathscr{L}}, 1_{\mathscr{L}}) = 0_{\mathscr{L}}$, then $\mathscr{U}$ is called conjunctive, if $\mathscr{U}(0_{\mathscr{L}}, 1_{\mathscr{L}}) = 1_{\mathscr{L}}$, then $\mathscr{U}$ is called disjunctive. Although all uninorms on the unit interval are either conjunctive or disjunctive [56], this is not the case anymore for uninorms on $\mathscr{L}^I$ [18].

Now we construct R- and S-implications derived from uninorms on $\mathscr{L}^I$.

Proposition 31. *[25] Let $\mathscr{U}$ be a uninorm on $\mathscr{L}^I$ with neutral element $e \in L^I \setminus \{0_{\mathscr{L}^I}, 1_{\mathscr{L}^I}\}$. Let $\Omega = \{\omega \mid \omega \in L^I$ and $\omega_2 > 0\}$. The mapping $\mathscr{I}_{\mathscr{U}} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathscr{I}_{\mathscr{U}}(x,y) = \sup\{z \mid z \in L^I \text{ and } \mathscr{U}(x,z) \leq_{L^I} y\}$$

is an implication on $\mathscr{L}^I$ if and only if

$$(\forall \omega \in \Omega)(\mathscr{U}(0_{\mathscr{L}^I}, \omega) = 0_{\mathscr{L}^I}).$$

As a consequence of this proposition, if $\mathscr{U}$ is conjunctive, then $\mathscr{I}_{\mathscr{U}}$ is an implication on $\mathscr{L}^I$. Note also that $\mathscr{I}_{\mathscr{U}}(e,x) = x$, for all $x \in L^I$.

Proposition 32. *[25] Let $\mathscr{U}$ be a uninorm and $\mathscr{N}$ a negation on $\mathscr{L}^I$. Then the mapping $\mathscr{I}_{\mathscr{U},\mathscr{N}} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathscr{I}_{\mathscr{U},\mathscr{N}}(x,y) = \mathscr{U}(\mathscr{N}(x),y)$$

is an implication on $\mathscr{L}^I$ if and only if $\mathscr{U}$ is disjunctive.

8 Conclusion

In this work we have listed some results pertaining to implications in interval-valued fuzzy set theory. We have described several possibilities to represent such implications using implications on the unit interval. We gave a characterization of the implications in interval-valued fuzzy set theory which satisfy the Smets-Magrez axioms. We discussed the solutions of a particular distributivity equation involving strict t-norms. We extended monoidal logic to the interval-valued fuzzy case and we gave a soundness and completeness theorem which is similar to the one existing for monoidal logic. Finally we discussed some other constructions of implications in interval-valued fuzzy set theory.

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