Nonlocal problems for superconductivity

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Domain

- Bounded domain $\Omega$ in $\mathbb{R}^3$ with a Lipschitz continuous boundary $\Gamma$       
- $\nu$ denotes the outward unit normal vector on $\Gamma$ 
- $\Omega$ is occupied by a superconductive material 
- This is a material, which loses all resistivity below a certain temperature $T_c$ 
- Figure: a magnet levitates above a ceramic superconductor cooled by liquid nitrogen

Full Maxwell's equations ($\delta = 1$) and quasi-static Maxwell's equations ($\delta = 0$) for linear materials

\[
\begin{align*}
\nabla \times \mathbf{H} & = J + \delta \varepsilon \partial_t \mathbf{E} \\
\nabla \times \mathbf{E} & = -\mu \partial_t \mathbf{H} \\
\n\nabla \cdot \mathbf{H} & = 0 \\
\n\mathbf{H} & \text{ magnetic field} \\
\mathbf{E} & \text{ electric field} \\
\mathbf{J} & \text{ current density} \\
\varepsilon & > 0 \quad \text{electric permittivity} \\
\mu & > 0 \quad \text{magnetic permeability}
\end{align*}
\]

Two-fluid model

\[
\begin{align*}
\mathbf{J} & = \mathbf{J}_s + \mathbf{J}_i \\
\mathbf{J}_s & = \sigma \mathbf{E} \quad \text{Ohm's law} \\
\mathbf{J}_i & = \mathbf{J}_i \quad \text{superconducting current density}
\end{align*}
\]

Local law for $\mathbf{J}_s$: London equations (1935)

\[
\begin{align*}
\partial_t \mathbf{J}_s & = \Lambda^{-1} \mathbf{E} \\
\nabla \times \mathbf{J}_s & = -\Lambda^{-1} \mathbf{B} \\
\Lambda & = \frac{m}{n_s q^2} \\
\n\Rightarrow \text{Correct description of both basic properties of superconductors: perfect conductivity and perfect diamagnetism (Meissner effect)}
\end{align*}
\]

\[
\nabla \cdot \mathbf{B} = 0 \rightarrow \exists \mathbf{A} \in H^1(\Omega) \text{ such that } \mathbf{B} = \nabla \times \mathbf{A} \text{ and } \nabla \cdot \mathbf{A} = 0
\]

\[
\nabla \times \mathbf{J}_s = -\Lambda^{-1} \mathbf{B} \\
\mathbf{J}_s(x, t) = -\Lambda^{-1} \mathbf{A}(x, t), \quad (x, t) \in \Omega \times (0, T)
\]

Lemma

\[
(x, t) \in \Omega \times (0, T), \quad \mathbf{H} \cdot \nu = 0 \text{ on } \Gamma \Rightarrow \nabla \times \mathbf{J}_s(x, t) = -\int_{\Omega} K(x - x') H(x', t) \, dx' = -\mathbf{K} \mathbf{H}(x, t) \text{ with } \mathbf{K} : \Omega \rightarrow \mathbb{R}^3 : \mathbf{y} \mapsto \left\{ \begin{array}{ll} \frac{c}{\rho} \exp \left( -\frac{|y|}{r_0} \right) & |y| < r_0; \\ 0 & |y| \geq r_0 \end{array} \right.
\]

Consequence: two models ($\delta = 0, 1$)

\[
\nabla \times \nabla \times \mathbf{H} = \sigma \nabla \times \mathbf{E} + \nabla \times \mathbf{J}_s + \delta \varepsilon \nabla \times \partial_t \mathbf{E} \quad \Rightarrow \quad \delta \varepsilon \mu \partial_t \mathbf{H} + \sigma \mu \partial_t \mathbf{H} - \Delta \mathbf{H} + \mathbf{K} \times \mathbf{H} = 0
\]

Solution method

- Time discretization is based on Backward Euler’s method 
- Rothe’s method \Rightarrow unique weak solution for both problems, numerical scheme

Further research

- Comparison of the two models, practical applications 
- Nonlinear model for $\mathbf{H}$

Numerical Experiment

- Penetration of magnetic field and induced current into the material (unit cube) 
- Main difficulty: programming convolution term in Fenics (not finished)

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