Imprecise multinomial processes

an overview of different approaches
and how they are related to each other

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13 November 2012
What is an imprecise multinomial process?

A sequence of random variables

\[ X_1, X_2, \ldots, X_n, \ldots \]

each assuming values in the same finite set

\[ \mathcal{X} = \{ H, T \} \]

\[ \{ 1, 2, 3, 4, 5, 6 \} \]
What is an imprecise multinomial process?

A sequence of random variables

\[ X_1, X_2, \ldots, X_n, \ldots \]

satisfying the IID property

INDEPENDENT
IDENTICALLY DISTRIBUTED
What is an imprecise multinomial process?

A sequence of random variables

$X_1, X_2, \ldots, X_n, \ldots$

satisfying the IID property

INDEPENDENT
IDENTICALLY DISTRIBUTED
Modelling a single variable
How to model a single random variable?

The precise approach: probability mass function / prevision

**probability mass function** $p$

$$\forall x \in \mathcal{X} \quad p(x) \geq 0$$

$$\sum_{x \in \mathcal{X}} p(x) = 1$$

**prevision** $P$ (expectation operator)

$$\forall f : \mathcal{X} \rightarrow \mathbb{R}$$

$$P(f) = \sum_{x \in \mathcal{X}} p(x)f(x)$$

$$P(f) \geq \min f$$

$$P(f_1 + f_2) = P(f_1) + P(f_1)$$

$$P(\lambda f) = \lambda P(f)$$
How to model a single random variable?

The **precise** approach: **probability mass function** / **prevision**

**probability mass function** $p$

$$\forall x \in \mathcal{X} \quad p(x) \geq 0$$

$$\sum_{x \in \mathcal{X}} p(x) = 1$$

**prevision** $P$ (expectation operator)

$$\forall f : \mathcal{X} \rightarrow \mathbb{R}$$

$$P(f) = \sum_{x \in \mathcal{X}} p(x)f(x)$$

**EXAMPLE:** $\mathcal{X} = \{H, T\}$

$p(H) = 4/10$

$p(T) = 6/10$

$P(f) = 4/10 \cdot f(H) + 6/10 \cdot f(T)$

$I_H(H) = 1, I_H(T) = 0$

$\Rightarrow P(I_H) = 4/10 = p(H)$

$f(H) = -1, f(T) = 3$

$\Rightarrow P(f) = 1.4$
How to model a single random variable?

An imprecise approach: credal set / coherent lower prevision [1]

credal set $\mathcal{M}$

Equivalent:

closed and convex set of probability mass functions

coherent lower prevision $\underline{P}$

$\forall \ f : \mathcal{X} \rightarrow \mathbb{R}$

$\underline{P}(f) = \min\{P(f) : p \in \mathcal{M}\}$

COHERENCE:

$\underline{P}(f) \geq \min f$

$\underline{P}(f_1 + f_2) \geq \underline{P}(f_1) + \underline{P}(f_1)$

$\underline{P}(\lambda f) = \lambda \underline{P}(f)$

$\geq 0$
How to model a single random variable?

An imprecise approach: credal set / coherent lower prevision \[1\]

credal set \( \mathcal{M} \)

- closed and convex set of probability mass functions

coherent lower prevision \( \underline{p} \)

\[\forall f : \mathcal{X} \rightarrow \mathbb{R}\]

\[\underline{p}(f) = \min\{p(f) : p \in \mathcal{M}\}\]

EXAMPLE: \( \mathcal{X} = \{H, T\} \)

\[p(H) = \theta \in [1/4, 1/2] \]

\[p(T) = 1 - \theta \]

\[\underline{p}(f) = \min\{\theta f(H) + (1-\theta)f(T)\} \quad \theta \in [1/4, 1/2]\]

\[I_H(H) = 1, \quad I_H(T) = 0\]

\[\underline{p}(I_H) = 1/4 = p(H)\]

\[f(H) = -1, \quad f(T) = 3\]

\[\underline{p}(f) = 1\]
How to model a single random variable?

An *imprecise* approach: credal set / coherent lower prevision [1]

Credal set $\mathcal{M}$

- Closed and convex
- Set of probability mass functions

Coherent lower prevision $\underline{P}$

$$\forall f : \mathcal{X} \rightarrow \mathbb{R}$$

$$\underline{P}(f) = \min\{P(f) : p \in \mathcal{M}\}$$

Coherent upper prevision $\overline{P}$

$$\forall f : \mathcal{X} \rightarrow \mathbb{R}$$

$$\overline{P}(f) = \max\{P(f) : p \in \mathcal{M}\}$$

EQUIVALENT
How to model a single random variable?

An imprecise approach: credal set / coherent lower prevision

Credal set $\mathcal{M}$

Closed and convex set of probability mass functions

Coherent lower prevision $\underline{P}$

$\forall f : \mathcal{X} \rightarrow \mathbb{R}$

$\underline{P}(f) = \min\{P(f) : p \in \mathcal{M}\}$

Coherent upper prevision $\overline{P}$

$\forall f : \mathcal{X} \rightarrow \mathbb{R}$

$\overline{P}(f) = \max\{P(f) : p \in \mathcal{M}\}$

Example: $\mathcal{X} = \{H, T\}$

$\underline{P}(I_H) = 1/4$, $\overline{P}(I_H) = 1/2$

$\underline{P}(f) = 1$, $\overline{P}(f) = 2$
How to model a single random variable?

An imprecise approach: credal set / coherent lower prevision

credal set $\mathcal{M}$

closed and convex set of probability mass functions

cohereent lower prevision $\underline{P}$

$\forall f : \mathcal{X} \to \mathbb{R}$

$\underline{P}(f) = \min\{P(f) : p \in \mathcal{M}\}$

EQUIVALENT

coherent upper prevision $\overline{P}$

$\forall f : \mathcal{X} \to \mathbb{R}$

$\overline{P}(f) = \max\{P(f) : p \in \mathcal{M}\}$

$\overline{P}(f) = \max\{-P(-f) : p \in \mathcal{M}\}$

$\overline{P}(f) = -\min\{-P(-f) : p \in \mathcal{M}\}$

$\overline{P}(f) = -\underline{P}(-f)$

We will focus on lower previsions!
How to model a single random variable?

An **imprecise** approach: **coherent set of desirable gambles**

A coherent set of desirable gambles \( \mathcal{D} \)

We model a **subject’s beliefs** about a variable by looking at the **gambles he is willing to accept** on its value

**EXAMPLE:** \( \mathcal{X} = \{H, T\} \)
How to model a single random variable?

An imprecise approach: coherent set of desirable gambles

A coherent set of desirable gambles $\mathcal{D}$

We model a subject’s beliefs about a variable by looking at the gambles he is willing to accept on its value

**COHERENCE:**

$$f \leq 0 \quad \Rightarrow \quad f \notin \mathcal{D}$$

**EXAMPLE:** $\mathcal{X} = \{H, T\}$
How to model a single random variable?

An *imprecise* approach: **coherent set of desirable gambles**

A coherent set of desirable gambles $\mathcal{D}$

We model a subject’s beliefs about a variable by looking at the gambles he is willing to accept on its value.

**COHERENCE:**

\[
\begin{align*}
 f \leq 0 & \Rightarrow f \notin \mathcal{D} \\
 f > 0 & \Rightarrow f \in \mathcal{D}
\end{align*}
\]
How to model a single random variable?

An **imprecise** approach: coherent set of desirable gambles

A coherent set of desirable gambles $\mathcal{D}$

We model a **subject’s beliefs** about a variable by looking at the **gambles he is willing to accept** on its value

**COHERENCE:**

\[
\begin{align*}
f \leq 0 & \implies f \notin \mathcal{D} \\
f > 0 & \implies f \in \mathcal{D} \\
f \in \mathcal{D} & \implies \lambda f \in \mathcal{D} \quad (\lambda > 0)
\end{align*}
\]
How to model a single random variable?

An imprecise approach: coherent set of desirable gambles

A coherent set of desirable gambles $\mathcal{D}$

We model a subject’s beliefs about a variable by looking at the gambles he is willing to accept on its value

**COHERENCE:**

\[
\begin{align*}
  f &\leq 0 \Rightarrow f \notin \mathcal{D} \\
  f &> 0 \Rightarrow f \in \mathcal{D} \\
  f \in \mathcal{D} \Rightarrow \lambda f \in \mathcal{D} \ (\lambda > 0) \\
  f_1, f_2 \in \mathcal{D} \Rightarrow f_1 + f_2 \in \mathcal{D}
\end{align*}
\]

**EXAMPLE:** $\mathcal{X} = \{H, T\}$
How to model a single random variable?

An imprecise approach: coherent set of desirable gambles

A coherent set of desirable gambles $\mathcal{D}$

We model a subject’s beliefs about a variable by looking at the gambles he is willing to accept on its value

EXAMPLE: $\mathcal{X} = \{H,T\}$
How to model a single random variable?

An **imprecise** approach: **coherent set of desirable gambles**

A coherent set of desirable gambles $\mathcal{D}$

We model a subject’s beliefs about a variable by looking at the gambles he is willing to accept on its value

$$f \in \mathcal{D}$$

**EXAMPLE:** $\mathcal{X} = \{H,T\}$

$f(H) = -1$

$f(T) = 3$
How to model a single random variable?

An imprecise approach: coherent set of desirable gambles

A coherent set of desirable gambles $\mathcal{D}$

We model a subject’s beliefs about a variable by looking at the gambles he is willing to accept on its value

Buying price $\mu$

$$f - \mu \in \mathcal{D}$$

Example: $\mathcal{X} = \{H,T\}$

$\mathcal{D}$

$f(H) = -1.5$

$f(T) = 2.5$
How to model a single random variable?

An imprecise approach: coherent set of desirable gambles

A coherent set of desirable gambles \( \mathcal{D} \)

We model a subject’s beliefs about a variable by looking at the gambles he is willing to accept on its value

supremum buying price

\[
\sup\{\mu : f - \mu \in \mathcal{D}\}
\]

EXAMPLE: \( \mathcal{X} = \{H,T\} \)

\[
\begin{align*}
f(H) &= -2 \\
f(T) &= 2
\end{align*}
\]
How to model a single random variable?

An imprecise approach: coherent set of desirable gambles

A coherent set of desirable gambles $\mathcal{D}$

We model a subject’s beliefs about a variable by looking at the gambles he is willing to accept on its value.

Coherent lower prevision $\underline{P}$

Supremum buying price

$\underline{P}(f) = \sup\{\mu : f - \mu \in \mathcal{D}\}$

Example: $\mathcal{X} = \{H, T\}$

$\underline{P}(f) = 1$
How to model a single random variable?

An *imprecise* approach: *coherent set of desirable gambles* [1]

A coherent set of desirable gambles \( \mathcal{D} \)

We model a subject’s beliefs about a variable by looking at the *gambles he is willing to accept* on its value

coherent upper prevision \( \underline{P} \)

infimum selling price

\[
\underline{P}(f) = \inf\{\mu : \mu - f \in \mathcal{D}\}
\]

**EXAMPLE:** \( \mathcal{X} = \{H, T\} \)

\[
\underline{P}(f) = 2
\]

\[
\overline{P}(f) = 1
\]

\[
f = 2 - f
\]
Precise multinomial process
The precise multinomial process

A sequence of random variables

\[ X_1, X_2, \ldots, X_n, \ldots \]

satisfying the IID property

INDEPENDENT

IDENTICALLY DISTRIBUTED
The precise multinomial process

A sequence of random variables

\[ X_1, X_2, \ldots, X_n, \ldots \]
The precise multinomial process

A sequence of random variables

\[ X_1, X_2, \ldots, X_n, \ldots \]

TIME CONSISTENCY

MARGINALISATION

\[ X_1, X_2, \ldots, X_n, \ldots, X_m, \ldots \]
The precise multinomial process

A sequence of random variables

$X_1, X_2, \ldots, X_n$
The precise multinominal process

A sequence of random variables

\[ X_1, X_2, X_3 \]
The precise multinomial process

\[ X_1, X_2, X_3 \]

\[ p_1 \cdot p_2 \cdot p_3 = p_{1,2,3} \text{  INDEPENDENT} \]

\[ p \quad p \quad p \quad \text{IDENTICALLY DISTRIBUTED} \]
The precise multinomial process

\[
X_1, X_2, X_3
\]

\[
p_1(x_1) \cdot p_2(x_2) \cdot p_3(x_3) = p_{1,2,3}(x_1, x_2, x_3)
\]

\[
p(x_1) \cdot p(x_2) \cdot p(x_3)
\]

\[
\forall f : \mathcal{X}^3 \rightarrow \mathbb{R}
\]

\[
P_{1,2,3}(f) = \sum_{x_1 \in \mathcal{X}} \sum_{x_2 \in \mathcal{X}} \sum_{x_3 \in \mathcal{X}} p_{1,2,3}(x_1, x_2, x_3)f(x_1, x_2, x_3)
\]
The precise multinomial process

\[
\begin{align*}
H & \quad T & \quad T \\
\| & \quad \| & \quad \| \\
X_1, X_2, X_3
\end{align*}
\]

\[
p_1(H) \cdot p_2(T) \cdot p_3(T) = p_{1,2,3}(H,T,T) = 0.144
\]

\[
A = \{(H,H,H),(H,T,T)\}
\]

\[
P_{1,2,3}(I_A) = p_{1,2,3}(H,H,H) + p_{1,2,3}(H,T,T) = 0.208
\]

**Example:** \(X = \{H, T\}\)

\[
p(H) = 4/10, \quad p(T) = 6/10
\]
Forward irrelevant multinomial process
The forward irrelevant multinomial process

An interpretation for the precise multinomial process

\[ X_1, \quad X_2, \quad X_3 \]
\[ p(X_1), \quad p(X_2), \quad p(X_3) \]
\[ \overset{\text{\(\|\)}}{\overset{\text{\(\|\)}}{\overset{\text{\(\|\)}}{p_1(X_1) \quad p_2(X_2) \quad p_3(X_3)}}} \]
\[ = p_{1,2,3}(X_1,X_2,X_3) \]

IDENTICALLY DISTRIBUTED
INDEPENDENT
The forward irrelevant multinomial process

An interpretation for the precise multinomial process

\[ X_1, \quad X_2, \quad X_3 \]

\[
p(X_1) \quad \parallel \quad p(X_2) \quad \parallel \quad p(X_3)
\]

\[
p_1(X_1) \quad \cdot \quad p_2(X_2) \quad \cdot \quad p_3(X_3)
\]

\[
p_1(X_1) \quad \cdot \quad p_2(X_2|X_1) \quad \cdot \quad p_3(X_3|X_1,X_2) = p_{1,2,3}(X_1,X_2,X_3)
\]

\[
= p_{1,2,3}(X_1,X_2,X_3)
\]

IDENTICALLY DISTRIBUTED

INDEPENDENT
### The forward irrelevant multinomial process

An **interpretation** for the precise multinomial process

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(X_1)$</td>
<td>$p(X_2)$</td>
<td>$p(X_3)$</td>
</tr>
<tr>
<td>$p_1(X_1)$</td>
<td>$p_2(X_2)$</td>
<td>$p_3(X_3)$</td>
</tr>
<tr>
<td>$p_1(X_1) \cdot p_2(X_2 \mid X_1) \cdot p_3(X_3 \mid X_1, X_2) = p_{1,2,3}(X_1, X_2, X_3)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The value of previous variables is **irrelevant** for our beliefs about the current one!
The forward irrelevant multinomial process

An interpretation for the precise multinomial process

\[ X_1, X_2, X_3 \]

\[ \begin{align*}
  & P( ) \quad P( ) \quad P( ) \\
  & \text{IDENTICALLY DISTRIBUTED} \\
  & \text{INDEPENDENT} \\
  & \text{II} \quad \text{II} \quad \text{II} \\
  & P_1( ) \quad P_2( ) \quad P_3( ) \\
  & \text{II} \quad \text{II} \quad \text{II} \\
  & P_1( ) \quad ? \quad P_2( |X_1) \quad ? \quad P_3( |X_1,X_2) \quad ? \quad P_{1,2,3}( ) \\
\end{align*} \]

The value of previous variables is irrelevant for our beliefs about the current one!
The forward irrelevant multinomial process

An interpretation for the precise multinomial process

\[ X_1, \quad X_2, \quad X_3 \]

\[ \begin{align*}
P_1(\ ) & \quad \quad \quad P_2(\ ) \quad \quad \quad P_3(\ ) \\
P_1(\ ) & \quad \quad \quad P_2(\ |X_1) \quad \quad \quad P_3(\ |X_1,X_2) \\
P_1(\ ) & \quad \quad \quad P_2(\ |X_1) \quad \quad \quad P_3(\ |X_1,X_2) \\
\end{align*} \]

\[ P_{1,2,3}( f(X_1,X_2,X_3) ) = P_1( P_2( P_3( f(X_1,X_2,X_3) |X_1,X_2 ) |X_1 ) ) \]

\[ = P_1( P_2( P_3( f(X_1,X_2,X_3) ) ) ) \]

Imprecise Bernoulli processes
The forward irrelevant multinomial process

Described using coherent lower previsions

\[ X_1, \quad X_2, \quad X_3 \]

\[ P( ) \quad P( ) \quad P( ) \]

\[ \equiv \quad \equiv \quad \equiv \]

\[ P_1( ) \quad P_2( ) \quad P_3( ) \]

\[ \equiv \quad \equiv \quad \equiv \]

\[ P_1( ) \quad P_2( |X_1) \quad P_3( |X_1,X_2) \quad P_{1,2,3}( ) \]

The value of previous variables is irrelevant for our beliefs about the current one!
The forward irrelevant multinomial process

Described using coherent lower previsions

\[ X_1, \ X_2, \ X_3 \]

\[ \underbrace{P(\ )}_1 \mapsto \underbrace{P(\ )}_1 \mapsto \underbrace{P(\ )}_1 \mapsto \]

\[ \underbrace{P(\ )}_2 \mapsto \underbrace{P(\ |X_1)}_2 \mapsto \underbrace{P(\ |X_1,X_2)}_3 \mapsto \]

\[ \overbrace{P_{1,2,3}(f(X_1,X_2,X_3))} = \underbrace{P(\ )}_1 \mapsto \underbrace{P(\ )}_2 \mapsto \underbrace{P(\ )}_3 \mapsto \]

\[ = P(\ P(\ P( f(X_1,X_2,X_3) | X_1,X_2 ) | X_1 ) ) ) \]
The forward irrelevant multinomial process

Described using coherent lower previsions

**EXAMPLE:** $\mathcal{X} = \{H, T\}$

$$P(\mathcal{f}) = \min\{\theta f(H) + (1-\theta) f(T)\}$$

$\theta \in [1/4, 1/2]$

$A = \{(H, H, H), (H, T, T)\}$

$P_{1,2,3}(I_A(X_1, X_2, X_3)) = ?$

$$P_{1,2,3}(f(X_1, X_2, X_3)) = P_1( P_2( P_3( f(X_1, X_2, X_3) | X_1, X_2 ) | X_1 ) )$$

$$P_3(I_A(H, H, X_3)) = 1/4$$

$P_3 I_A(H, H, H) = 1$

$P_3 I_A(H, H, T) = 0$
The forward irrelevant multinomial process

Described using coherent lower previsions

**EXAMPLE:** \( \mathcal{X} = \{H,T\} \)

\[
P(f) = \min\{\theta f(H) + (1-\theta)f(T)\} \\
\theta \in [1/4, 1/2]
\]

\( A = \{(H,H,H),(H,T,T)\} \)

\[
P_{1,2,3}( I_A(X_1,X_2,X_3) ) = ?
\]

\[
P_{1,2,3}( f(X_1,X_2,X_3) ) = P_1( P_2( P_3( f(X_1,X_2,X_3) | X_1,X_2 ) | X_1 ) ) \\
= P( P( P( f(X_1,X_2,X_3) ) ) )
\]

\[
P_3( I_A(H,H,X_3) ) = 1/4 \\
P_3( I_A(H,T,X_3) ) = 1/2 \\
P_3( I_A(T,H,X_3) ) = 0 \\
P_3( I_A(T,T,X_3) ) = 0
\]
The forward irrelevant multinomial process

Described using coherent lower previsions

**EXAMPLE:** \( \mathcal{X} = \{H, T\} \)

\[ P(f) = \min\{\theta f(H) + (1-\theta) f(T)\} \]
\[ \theta \in [1/4, 1/2] \]

\( A = \{(H,H,H), (H,T,T)\} \)

\[ P_{1,2,3}( I_A(X_1, X_2, X_3) ) = ? \]

\[ P_{1,2,3}( f(X_1, X_2, X_3) ) = P_1( P_2( P_3( f(X_1, X_2, X_3) | X_1, X_2 ) | X_1 ) ) \]
\[ = P( P( P( f(X_1, X_2, X_3) ) ) ) \]

\[ P_3( I_A(H,H,X_3) ) = 1/4 \]
\[ P_3( I_A(H,T,X_3) ) = 1/2 \]
\[ P_2( P_3( I_A(H,X_2,X_3) ) ) = 3/8 \]
The forward irrelevant multinomial process

Described using coherent lower previsions

EXAMPLE: \( \mathcal{H} = \{H, T\} \)

\[
P(f) = \min \{\theta f(H) + (1-\theta)f(T)\} \\
\theta \in [1/4, 1/2]
\]

\( A = \{(H,H,H),(H,T,T)\} \)

\[
P_{1,2,3}(I_A(X_1,X_2,X_3)) = ?
\]

\[
P_1,2,3(f(X_1,X_2,X_3)) = P_1\left( P_2\left( P_3\left( f(X_1,X_2,X_3) \mid X_1,X_2 \mid X_1 \right) \right) \right) = P\left( P\left( P\left( f(X_1,X_2,X_3) \right) \right) \right)
\]
The forward irrelevant multinomial process

Described using coherent lower previsions

**EXAMPLE:** $\mathcal{X} = \{H,T\}$

\[
\begin{align*}
P(f) &= \min\{\theta f(H) + (1-\theta)f(T)\} \\
\theta &\in [1/4, 1/2] \\
A &= \{(H,H,H),(H,T,T)\}
\end{align*}
\]

\[
\begin{align*}
P_{1,2,3}(\mathcal{I}_A(H,X_2,X_3)) &= 3/32 = P_1(\underbrace{P_2(\underbrace{P_3(\mathcal{I}_A(X_1,H_2,X_3))}_{\text{PB}})}_{\text{PB}})
\end{align*}
\]

\[
\begin{align*}
P_{1,2,3}(\mathcal{I}_A(T,X_2,X_3)) &= 0 \\
P_{2,3}(\mathcal{I}_A(H,X_2,X_3)) &= 3/8
\end{align*}
\]
The forward irrelevant multinomial process

Described using coherent lower previsions

**EXAMPLE:** $\mathcal{X} = \{H, T\}$

$$P(f) = \min\{\theta f(H) + (1-\theta)f(T)\}$$

$\theta \in [1/4, 1/2]$

$A = \{(H,H,H), (H,T,T)\}$

$$P_{1,2,3}(I_A(X_1, X_2, X_3)) = \frac{3}{32}$$

$$\overline{P}_{1,2,3}(I_A(X_1, X_2, X_3)) = -P_{1,2,3}(-I_A(X_1, X_2, X_3)) = \frac{11}{32}$$
Independent multinomial process
The independent multinomial process

An interpretation for the precise multinomial process

\[ X_1, \quad X_2, \quad X_3 \]

\[ \begin{align*}
  & p(X_1) \\
  \parallel & p_1(X_1) \\
  & p(X_2) \\
  \parallel & p_2(X_2) \\
  & p(X_3) \\
  \parallel & p_3(X_3)
\end{align*} \]

\[ \text{IDENTICALLY DISTRIBUTED} \]

\[ \text{INDEPENDENT} \]

\[ = \quad p_{1,2,3}(X_1,X_2,X_3) \]
The independent multinomial process

An interpretation for the precise multinomial process

\[ X_1, \quad X_2, \quad X_3 \]

\[ p(X_1) \quad \| \quad p(X_2) \quad \| \quad p(X_3) \]

\[ p_1(X_1) \quad \| \quad p_2(X_2 | X_1) \quad \| \quad p_3(X_3 | X_1, X_2) \]

The value of previous variables is irrelevant for our beliefs about the current one!
The independent multinomial process

An interpretation for the precise multinomial process

The value of previous and future variables is irrelevant for our beliefs about the current one!
The independent multinomomial process

An interpretation for the precise multinomomial process

\[ X_1, X_2, X_3 \]

\[ P(X_1), P(X_2), P(X_3) \]

\[ \text{IDENTICALLY DISTRIBUTED} \]

\[ \text{INDEPENDENT} \]

\[ P_1(X_1), P_2(X_2), P_3(X_3) \]

\[ P_1(X_1|X_2, X_3), P_2(X_2|X_1, X_3), P_3(X_3|X_1, X_2) \]

The value of previous and future variables is irrelevant for our beliefs about the current one!
The independent multinominal process

Described using coherent lower previsions

The value of previous and future variables is irrelevant for our beliefs about the current one!
The independent multinomial process

Described using coherent sets of desirable gambles

\[ X_1, X_2, X_3 \]

\( D_{1} \parallel D_{2} \parallel D_{3} \)

\( D_{1} | x_2, x_3 \parallel D_{2} | x_1, x_3 \parallel D_{3} | x_1, x_2 \)

 IDENTICALLY DISTRIBUTED

 EPISTEMICALLY INDEPENDENT

The value of previous and future variables is irrelevant for our beliefs about the current one!
The independent multinomial process

Described using **coherent sets of desirable gambles**

\[ X_1, \quad X_2, \quad X_3 \]

\[ \mathcal{D}_1 \parallel \mathcal{D}_{1|X_2,X_3} \]

\[ \mathcal{D}_2 \parallel \mathcal{D}_{2|X_1,X_3} \]

\[ \mathcal{D}_3 \parallel \mathcal{D}_{3|H,H} \]

\[ \mathcal{D}_{1,2,3} \]

\[ \mathcal{I}_{\{X_1=H\}} \cdot \mathcal{I}_{\{X_2=H\}} \cdot \mathcal{f}_{3|H,H} \]

**Identically Distributed**

**Epistemically Independent**
The independent multinomial process

Described using **coherent sets of desirable gambles** [4]

\[
\mathbf{f} \in \mathcal{D}_{1,2,3}
\]

\[
\iff \mathbf{f} = \sum_{x_2 \in \mathcal{X}} \sum_{x_3 \in \mathcal{X}} \mathbf{f}_{1 \mid x_2, x_3} \cdot \mathbb{I}_{\{x_2 = x_2\}} \cdot \mathbb{I}_{\{x_3 = x_3\}}
\]

\[
+ \sum_{x_1 \in \mathcal{X}} \sum_{x_3 \in \mathcal{X}} \mathbb{I}_{\{x_1 = x_1\}} \cdot \mathbf{f}_{2 \mid x_1, x_3} \cdot \mathbb{I}_{\{x_3 = x_3\}}
\]

\[
+ \sum_{x_1 \in \mathcal{X}} \sum_{x_2 \in \mathcal{X}} \mathbb{I}_{\{x_1 = x_1\}} \cdot \mathbb{I}_{\{x_2 = x_2\}} \cdot \mathbf{f}_{3 \mid x_1, x_2}
\]

IDENTICALLY DISTRIBUTED

EPISTEMICALLY INDEPENDENT

\[\mathbf{f} \in \mathcal{D}\]
The independent multinomial process

Described using coherent sets of desirable gambles

\[ f \in D_{1,2,3} \]

\[ f = \sum_{x_2 \in \mathcal{X}} \sum_{x_3 \in \mathcal{X}} \mathbf{1}_{\{x_2=x_2\}} \cdot f_1 | x_2, x_3 \cdot \mathbf{1}_{\{x_3=x_3\}} + \sum_{x_1 \in \mathcal{X}} \sum_{x_3 \in \mathcal{X}} \mathbf{1}_{\{x_1=x_1\}} \cdot f_2 | x_1, x_3 \cdot \mathbf{1}_{\{x_3=x_3\}} + \sum_{x_1 \in \mathcal{X}} \sum_{x_2 \in \mathcal{X}} \mathbf{1}_{\{x_1=x_1\}} \cdot \mathbf{1}_{\{x_2=x_2\}} \cdot f_3 | x_1, x_2 \]

\[ P_{1,2,3}(f) = \sup\{\mu : f - \mu \in D_{1,2,3}\} \]
The independent multinomial process

Described using coherent sets of desirable gambles

**EXAMPLE:** \( \mathcal{X} = \{H,T\} \)

\[ A = \{(H,H,H),(H,T,T)\} \]

\[ \mathbb{P}_{1,2,3}(I_A) = \sup\{\mu : I_A - \mu \in \mathcal{D}_{1,2,3}\} = 1/10 \]
The independent multinomial process

Described using coherent sets of desirable gambles

**EXAMPLE:** $\mathcal{X} = \{H, T\}$

$A = \{(H,H,H),(H,T,T)\}$

$\overline{P}_{1,2,3}(I_A) = \inf\{\mu : I_A - \mu \in D_{1,2,3}\} = 1/3$

$\underline{P}_{1,2,3}(I_A) = \sup\{\mu : I_A - \mu \in D_{1,2,3}\} = 1/10$
Permutability
Permutability

Consider any permutation $\pi$ of the set of indices $\{1, 2, 3\}$

Symmetry of the precise multinomial process

\[
p_{1,2,3}(X_1, X_2, X_3) = p_{1,2,3}(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)})
\]

\[
P_{1,2,3}(f(X_1, X_2, X_3)) = P_{1,2,3}(f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}))
\]
**Permutability**

**Permutability** of the imprecise multinomial process

Consider any permutation $\pi$ of the set of indices $\{1, 2, 3\}$

$$P_{1,2,3}(f(X_1, X_2, X_3)) = P_{1,2,3}(f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}))$$

$$f(X_1, X_2, X_3) \in D_{1,2,3} \iff f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}) \in D_{1,2,3}$$

**Symmetry** of the precise multinomial process

$$p_{1,2,3}(X_1, X_2, X_3) = p_{1,2,3}(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)})$$

$$P_{1,2,3}(f(X_1, X_2, X_3)) = P_{1,2,3}(f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}))$$
Permutability

Permutability of the imprecise multinomial process

Consider any permutation $\pi$ of the set of indices $\{1, 2, 3\}$

$$P_{1,2,3}(f(X_1, X_2, X_3)) = P_{1,2,3}(f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}))$$

$$f(X_1, X_2, X_3) \in D_{1,2,3} \iff f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}) \in D_{1,2,3}$$

The forward irrelevant multinomial process becomes equivalent with the independent multinomial process if we additionally impose permutability as a required property!
Strong multinomial process
The strong multinomial process

An interpretation for the precise multinomial process

\[ X_1, X_2, X_3 \quad \text{IDENTICALLY DISTRIBUTED} \]

\[ p_1 \cdot p_2 \cdot p_3 = \text{p}_{1,2,3} \quad \text{INDEPENDENT} \]
The strong multinomial process

An interpretation for the precise multinomial process

$$X_1, X_2, X_3$$

$$\{p_1\} \cdot \{p_2\} \cdot \{p_3\} = \{p_{1,2,3}\}$$

IDENTICALLY DISTRIBUTED

INDEPENDENT
The strong multinomial process

Described using credal sets

\[ X_1, X_2, X_3 \]
\[ p_1 \cdot p_2 \cdot p_3 = p_{1,2,3} \]
\[ \mathcal{M} \cdot \mathcal{M} \cdot \mathcal{M} = \mathcal{M}_{1,2,3} \]

IDENTICALLY DISTRIBUTED
(STRONGLY)
INDEPENDENT
TAKE CONVEX CLOSURE!
The strong multinomial process

Described using credal sets / coherent lower previsions

\[ p_1 \cdot p_2 \cdot p_3 = p_{1,2,3} \]

\[ M \quad M \quad M \]

\[ \mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M}_{1,2,3} \]

\[ P_{1,2,3}(f(X_1, X_2, X_3)) = \min\{P_{1,2,3}(f(X_1, X_2, X_3)) : p_{1,2,3} \in \mathcal{M}_{1,2,3} \} \]
The strong multinomial process

Described using credal sets / coherent lower previsions

EXAMPLE: \( \mathcal{H} = \{H, T\} \)

\[ \mathcal{M} = \{ p : p(H) = \theta \in [1/4, 1/2], p(T) = 1-\theta \} \]
\[ A = \{(H,H,H),(H,T,T)\} \]
\[ \underline{P}_{1,2,3}( I_A(X_1,X_2,X_3) ) = \min \{ \theta_1(\theta_2\theta_3+(1-\theta_2)(1-\theta_3)) = 1/8 \}
\]
\[ \theta_1 \in [1/4, 1/2] \]
\[ \theta_2 \in [1/4, 1/2] \]
\[ \theta_3 \in [1/4, 1/2] \]

\[ \underline{P}_{1,2,3}( f(X_1,X_2,X_3) ) = \min \{ \underline{P}_{1,2,3}( f(X_1,X_2,X_3) ) : p_{1,2,3} \in \mathcal{M}_{1,2,3} \} \]
The strong multinomial process

Described using **credal sets / coherent lower previsions**

**EXAMPLE:** $\mathcal{X} = \{\text{H, T}\}$

$\mathcal{M} = \{ p : p(\text{H}) = \theta \in [1/4, 1/2], p(\text{T}) = 1-\theta \}$

$A = \{ (\text{H,H,H}), (\text{H,T,T}) \}$

$P_{1,2,3}( I_A(X_1, X_2, X_3) ) = \min \{ \theta_1(\theta_2\theta_3 + (1-\theta_2)(1-\theta_3)) : \theta_1 \in [1/4, 1/2], \theta_2 \in [1/4, 1/2], \theta_3 \in [1/4, 1/2] \} = 1/8$

$\bar{P}_{1,2,3}( I_A(X_1, X_2, X_3) ) = \max \{ \theta_1(\theta_2\theta_3 + (1-\theta_2)(1-\theta_3)) : \theta_1 \in [1/4, 1/2], \theta_2 \in [1/4, 1/2], \theta_3 \in [1/4, 1/2] \} = 5/16$
Exchangeable multinomial process
The exchangeable multinomial process

An interpretation for the precise multinomial process

\[ X_1, X_2, X_3 \text{ IDENTICALLY DISTRIBUTED} \]

\[ \begin{align*}
&\ p_1 \cdot p_2 \cdot p_3 = p_{1,2,3} \\
&\ \| \quad \| \quad \| \\
&\ p \quad p \quad p \\
\end{align*} \text{ INDEPENDENT} \]
The exchangeable multinomial process

An interpretation for the precise multinomial process

\[ X_1, X_2, X_3 \]

\[ \begin{align*}
    p_1 &\cdot p_2 \cdot p_3 &= p_{1,2,3} \\
    \mathbb{I} &\mathbb{I} &\mathbb{I} \\
    p &= p &= p \\
    \mathbb{M}\{p\} &\text{INDEPENDENT}
\end{align*} \]
The exchangeable multinomial process

Described using credal sets

\[ X_1, X_2, X_3 \]

\[
\begin{align*}
\pi_1 \cdot \pi_2 \cdot \pi_3 &= \pi_{1,2,3} \\
\pi &= \pi_{1,2,3} \\
\mathcal{M} &= \mathcal{M}_{1,2,3}
\end{align*}
\]

IDENTICALLY DISTRIBUTED

INDEPENDENT

(Sensitivity analysis)

TAKE CONVEX CLOSURE!
The exchangeable multinomial process

Described using credal sets / coherent lower previsions

\[ X_1, X_2, X_3 \]

\[ p_1 \cdot p_2 \cdot p_3 = p_{1,2,3} \quad \text{IDENTICALLY DISTRIBUTED} \]

\[ p = p = p \]

\[ m \quad \mathcal{M}_{1,2,3} \quad \text{INDEPENDENT} \]

\[ (\text{Sensitivity analysis}) \]

\[ \mathcal{M} \quad \text{TAKE CONVEX CLOSURE!} \]

\[ p_{1,2,3}( f(X_1, X_2, X_3) ) = \min \{ p_{1,2,3}( f(X_1, X_2, X_3) ) : p_{1,2,3} \in \mathcal{M}_{1,2,3} \} \]
The exchangeable multinomial process

Described using credal sets / coherent lower previsions

**EXAMPLE:** \( \mathcal{X} = \{H,T\} \)

\[ M = \{ p : p(H) = \theta \in [1/4, 1/2], p(T) = 1-\theta \} \]

\[ A = \{(H,H,H),(H,T,T)\} \]

\[ P_{1,2,3}(I_A(X_1,X_2,X_3)) = \min_{\theta \in [1/4, 1/2]} \{ \theta(\theta^2 + (1-\theta)^2) = 5/32 \} \]

\[ P_{1,2,3}(f(X_1,X_2,X_3)) = \min\{ P_{1,2,3}(f(X_1,X_2,X_3)) : p_{1,2,3} \in M_{1,2,3} \} \]
The exchangeable multinomial process

Described using credal sets / coherent lower previsions

Example: \( \mathcal{H} = \{H,T\} \)

\[ \mathcal{M} = \{ p : p(H) = \theta \in [1/4, 1/2], p(T) = 1-\theta \} \]

\[ A = \{(H,H,H),(H,T,T)\} \]

\[ P_{1,2,3}( I_A(X_1,X_2,X_3) ) = \min_{\theta \in [1/4, 1/2]} \{\theta(\theta^2+(1-\theta)^2) = 5/32 \} \]

\[ P_{1,2,3}( I_A(X_1,X_2,X_3) ) = \max_{\theta \in [1/4, 1/2]} \{\theta(\theta^2+(1-\theta)^2) = 1/4 \} \]
An overview
An overview of the different approaches

**Example:** \( \mathcal{X} = \{H, T\} \)

\[ A = \{(H,H,H),(H,T,T)\} \]

**Local models**

Precise: \( p(H) = 4/10, \ p(T) = 6/10 \)

Imprecise: \( \mathcal{M} = \{ p : p(H) = \theta \in [1/4, 1/2], p(T) = 1-\theta \} \)

**Multinomial processes**

\[
\begin{align*}
P_{1,2,3}( I_A(X_1, X_2, X_3) ) & \quad \bar{P}_{1,2,3}( I_A(X_1, X_2, X_3) ) \\
\text{Precise:} & \quad 2496/12000 \quad 2496/12000 \\
\text{Forward irrelevant:} & \quad 1125/12000 \quad 4125/12000 \\
\text{Independent:} & \quad 1200/12000 \quad 4000/12000 \\
\text{Strong:} & \quad 1500/12000 \quad 3750/12000 \\
\text{Exchangeable:} & \quad 1875/12000 \quad 3000/12000 
\end{align*}
\]
Exchangeability
Consider any permutation $\pi$ of the set of indices $\{1, 2, 3\}$

**Symmetry** of the precise multinomial process

\[
p_{1,2,3}(X_1, X_2, X_3) = p_{1,2,3}(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)})
\]

\[
P_{1,2,3}( f(X_1, X_2, X_3) ) = P_{1,2,3}( f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}) )
\]

\[
P_{1,2,3}( f(X_1, X_2, X_3) - f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}) ) = 0
\]
Exchangeability

Exchangeability of the imprecise multinomial process

Consider any permutation \( \pi \) of the set of indices \( \{1, 2, 3\} \)

\[
P_{1,2,3}( f(X_1, X_2, X_3) - f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}) ) \geq 0
\]

Symmetry of the precise multinomial process

\[
p_{1,2,3}(X_1, X_2, X_3) = p_{1,2,3}(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)})
\]

\[
P_{1,2,3}( f(X_1, X_2, X_3) ) = P_{1,2,3}( f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}) )
\]

\[
P_{1,2,3}( f(X_1, X_2, X_3) - f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}) ) = 0
\]
**Exchangeability**

Exchangeability of the imprecise multinomial process

Consider any permutation $\pi$ of the set of indices $\{1, 2, 3\}$

$$P_{1,2,3}( f(X_1, X_2, X_3) - f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}) ) \geq 0$$

**MAIN RESULT:**

All four imprecise multinomial processes become **equivalent** with the **exchangeable** multinomial process if we additionally impose **exchangeability** (for all finite sequences) and **time consistency** as required properties!
References


