Triangular norms which are meet-morphisms in interval-valued fuzzy set theory

Glad Deschrijver
Fuzziness and Uncertainty Modelling Research Unit
Department of Applied Mathematics and Computer Science, Ghent University
Krijgslaan 281 (S9), B–9000 Gent, Belgium
E-mail: Glad.Deschrijver@UGent.be
Homepage: http://www.fuzzy.UGent.be

Abstract

In this paper we study t-norms on the lattice of closed subintervals of the unit interval. Unlike for t-norms on a product lattice for which there exists a straightforward characterization of t-norms which are join-morphisms, respectively meet-morphisms, the situation is more complicated for t-norms in interval-valued fuzzy set theory. In previous papers several characterizations were given of t-norms in interval-valued fuzzy set theory which are join-morphisms and which satisfy additional properties, but little attention has been paid to meet-morphisms. Therefore, in this paper, we focus on t-norms which are meet-morphisms. We consider a general class of t-norms and investigate under which conditions t-norms belonging to this class are meet-morphisms. We also characterize the t-norms which are both a join- and a meet-morphism and which satisfy an additional border condition.

Keywords: interval-valued fuzzy set, t-norm, meet-morphism

1 Introduction

Interval-valued fuzzy set theory [11, 15] is an extension of fuzzy theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. Another extension of fuzzy set theory is intuitionistic fuzzy set theory introduced by Atanassov [1]. In [8] it is shown that the underlying lattice of intuitionistic fuzzy set theory is isomorphic to the underlying lattice $\mathcal{L}_I$ of interval-valued fuzzy set theory.

In [6, 7, 5, 18] several characterizations of t-norms on $\mathcal{L}_I$ in terms of t-norms on the unit interval are given. In [13, 19, 20] t-norms on related and more general lattices are investigated. However all the characterizations in these papers only deal with t-norms which are join-morphisms. Unlike for t-norms on a product lattice for which there exists a straightforward characterization of t-norms which are join-morphisms [3], respectively meet-morphisms, the situation is more complicated for t-norms in interval-valued fuzzy set theory. Therefore, in this paper, we focus on t-norms which are meet-morphisms. We consider a general class of t-norms (given in [7]) and investigate under which conditions t-norms belonging to this class are meet-morphisms.
2 The lattice $\mathcal{L}^I$

Definition 2.1 We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where

$$L^I = \{ [x_1, x_2] \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 \leq x_2 \},$$

$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2)$, for all $[x_1, x_2], [y_1, y_2]$ in $L^I$.

Similarly as Lemma 2.1 in [8] it can be shown that $\mathcal{L}^I$ is a complete lattice.


Definition 2.3 [1] An intuitionistic fuzzy set on $U$ is a set

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

where $\mu_A(u) \in [0, 1]$ denotes the membership degree and $\nu_A(u) \in [0, 1]$ the non-membership degree of $u$ in $A$ and where for all $u \in U$, $\mu_A(u) + \nu_A(u) \leq 1$.

An intuitionistic fuzzy set $A$ on $U$ can be represented by the $\mathcal{L}^I$-fuzzy set $A$ given by

$$A : U \rightarrow L^I : u \mapsto [\mu_A(u), 1 - \nu_A(u)].$$

In Figure 1 the set $L^I$ is shown. Note that to each element $x = [x_1, x_2]$ of $L^I$ corresponds a point $(x_1, x_2) \in \mathbb{R}^2$.

![Figure 1: The grey area is $L^I$.](image)

In the sequel, if $x \in L^I$, then we denote its bounds by $x_1$ and $x_2$, i.e. $x = [x_1, x_2]$. The length $x_2 - x_1$ of the interval $x \in L^I$ is called the degree of uncertainty and is denoted by $x_\pi$.

The smallest and the largest element of $L^I$ are given by $0_{L^I} = [0, 0]$ and $1_{L^I} = [1, 1]$. Note that, for $x, y$ in $L^I$, $x <_{L^I} y$ is equivalent to $x \leq_{L^I} y$ and $x \neq y$, i.e. either $x_1 < y_1$ and $x_2 \leq y_2$, or $x_1 \leq y_1$ and $x_2 < y_2$. We define for further usage the set $D = \{ [x_1, x_1] \mid x_1 \in [0, 1] \}$.

Note that for any non-empty subset $A$ of $L^I$ it holds that

$$\sup A = \{ \sup \{x_1 \mid [x_1, x_2] \in A\}, \sup \{x_2 \mid [x_1, x_2] \in A\}\},$$

$$\inf A = \{ \inf \{x_1 \mid [x_1, x_2] \in A\}, \inf \{x_2 \mid [x_1, x_2] \in A\}\}.$$
**Theorem 2.1 (Characterization of supremum in $L^I$)** \[8\] Let $A$ be an arbitrary non-empty subset of $L^I$ and $a \in L^I$. Then $a = \sup A$ if and only if
\[
(\forall x \in A)(x \leq_L^I a) \\
and (\forall \varepsilon > 0)(\exists z \in A)(z > a - \varepsilon)
\]
and $(\forall \varepsilon > 0)(\exists z \in A)(z > a - \varepsilon)$.

**Definition 2.4** A $t$-norm on $L^I$ is a commutative, associative, increasing mapping $T : (L^I)^2 \to L^I$ which satisfies $T(1_{L^I}, x) = x$, for all $x \in L^I$.

**Example 2.1** \[7\] We give some special classes of $t$-norms on $L^I$. Let $T, T_1$ and $T_2$ be $t$-norms on $([0, 1], \leq)$ such that $T_1(x_1, y_1) \leq T_2(x_1, y_1)$ for all $x_1, y_1$ in $[0, 1]$, and let $t \in [0, 1]$. Then we have the following classes:

- **t-representable $t$-nормs:**
  \[T_{T_1, T_2}(x, y) = [T_1(x_1, y_1), T_2(x_2, y_2)],\]
  for all $x, y$ in $L^I$;

- **pseudo-$t$-representable $t$-nормs:**
  \[T_T(x, y) = [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))],\]
  for all $x, y$ in $L^I$;

- **$T_{T, t}$:**
  \[T_{T, t}(x, y) = [T(t_1, y_1), \max(T(t_1, x_2, y_2), T(t_2, x_2, y_1))],\]
  for all $x, y$ in $L^I$;

- **$T_T^t(x, y)$:**
  \[T_T^t(x, y) = [\min(T(x_1, y_2), T(x_2, y_2)), T(x_2, y_2)],\]
  for all $x, y$ in $L^I$;

- **$T_{T_1, T_2, t}(x, y)$:**
  \[T_{T_1, T_2, t}(x, y) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(x_2, y_1))],\]
  for all $x, y$ in $L^I$, where $T_1$ and $T_2$ additionally satisfy, for all $x_1, y_1$ in $[0, 1],$
  \[T_2(x_1, y_1) > T_2(t, T_2(x_1, y_1)) \Rightarrow T_1(x_1, y_1) = T_2(x_1, y_1). \tag{1}\]

In Theorem 5 of \[7\] it is shown that $T_{T_1, T_2, t}$ is indeed a $t$-norm on $L^I$ if $T_1$ and $T_2$ satisfy \[1, 7\].

**Definition 2.5** We say that a $t$-norm $T$ on $L^I$ is

- **a join-morphism** if for all $x, y, z$ in $L^I$,
  \[T(x, \sup(y, z)) = \sup(T(x, y), T(x, z));\]

- **a meet-morphism** if for all $x, y, z$ in $L^I$,
  \[T(x, \inf(y, z)) = \inf(T(x, y), T(x, z));\]

\[1\] Note that the condition in Theorem 5 of \[7\] that $T_1$ and $T_2$ are left-continuous is not used to prove that $T_{T_1, T_2, t}$ is a $t$-norm.
• a sup-morphism if for all $x \in L^I$ and $\emptyset \neq Z \subseteq L^I$,
  \[ T(x, \sup Z) = \sup\{T(x, z) \mid z \in Z\}; \]

• an inf-morphism if for all $x \in L^I$ and $\emptyset \neq Z \subseteq L^I$,
  \[ T(x, \inf Z) = \inf\{T(x, z) \mid z \in Z\}. \]

**Definition 2.6** Let $n \in \mathbb{N} \setminus \{0\}$. If for an $n$-ary mapping $f$ on $[0, 1]$ and an $n$-ary mapping $F$ on $L^I$ it holds that

\[ F([a_1, a_1], \ldots, [a_n, a_n]) = [f(a_1, \ldots, a_n), f(a_1, \ldots, a_n)], \]

for all $(a_1, \ldots, a_n) \in [0, 1]^n$, then we say that $F$ is a natural extension of $f$ to $L^I$.

Clearly, for any mapping $F$ on $L^I$, $F(D, \ldots, D) \subseteq D$ if and only if there exists a mapping $f$ on $[0, 1]$ such that $F$ is a natural extension of $f$ to $L^I$. E.g. $T_{T,T}, T_T, T_{T,t} = T_{T,T,t}$ and $T_T'$ are all natural extensions of $T$ to $L^I$, $N_s$ is a natural extension of $N_s$.

**Example 2.2** Let, for all $x, y$ in $[0, 1]$,

\[ T_W(x, y) = \max(0, x + y - 1), \]

\[ T_P(x, y) = xy, \]

\[ T_D(x, y) = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1, \\ 0, & \text{else}. \end{cases} \]

Then $T_W$, $T_P$ and $T_D$ are t-norms on $([0, 1], \leq)$. Let now, for all $x, y$ in $L^I$,

\[ T_W(x, y) = [\max(0, x_1 + y_1 - 1), \max(0, x_1 + y_2 - 1, x_2 + y_1 - 1)], \]

\[ T_P(x, y) = [x_1y_1, \max(x_1y_2, x_2y_1)]. \]

Then $T_W$ and $T_P$ are t-norms on $L^I$. Furthermore, $T_W$ and $T_P$ are natural extensions of $T_W$ and $T_P$ respectively.

We will also need the following result and definition (see [2, 12, 14, 16, 17]).

**Theorem 2.2** Let $\{T_\alpha\}_{\alpha \in A}$ be a family of t-norms and $\{[a_\alpha, e_\alpha]\}_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0, 1]$. Then the function $T : [0, 1]^2 \to [0, 1]$ defined by, for all $x, y$ in $[0, 1]$,

\[ T(x, y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha \left( \frac{x - a_\alpha}{e_\alpha - a_\alpha}, \frac{y - a_\alpha}{e_\alpha - a_\alpha} \right), & \text{if } (x, y) \in [a_\alpha, e_\alpha]^2; \\ \min(x, y), & \text{otherwise}, \end{cases} \]  

(2)

is a t-norm on $([0, 1], \leq)$.

**Definition 2.7** Let $\{T_\alpha\}_{\alpha \in A}$ be a family of t-norms and $\{[a_\alpha, e_\alpha]\}_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0, 1]$. The t-norm $T$ defined by (2) is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, T_\alpha \rangle$, $\alpha \in A$, and we will write

\[ T = ((a_\alpha, e_\alpha, T_\alpha))_{\alpha \in A}. \]
Let $A$ be an arbitrary countable index-set and $T_\alpha$ a t-norm on $L^I$, for all $\alpha \in A$. Define, for all $\alpha \in A$ and for all $a_\alpha, e_\alpha$ in $D$ with $a_\alpha \leq_L e_\alpha$, the following sets and mappings:

\[
J_\alpha = \{x \mid x \in L^I \text{ and } a_\alpha \leq_L x \leq_L e_\alpha\};
\]

\[
J_\alpha^* = \{x \mid x \in L^I \text{ and } x_1 > (a_\alpha)_1 \text{ and } x_2 \leq (e_\alpha)_2\};
\]

\[
\Phi_\alpha : J_\alpha \rightarrow L^I : x \mapsto \left[\frac{x_1 - (a_\alpha)_1}{(e_\alpha)_1 - (a_\alpha)_1}, \frac{x_2 - (a_\alpha)_2}{(e_\alpha)_2 - (a_\alpha)_2}\right], \forall x \in J_\alpha;
\]

\[
\Phi_\alpha^{-1} : L^I \rightarrow J_\alpha : x \mapsto \{(a_\alpha)_1 + x_1((e_\alpha)_1 - (a_\alpha)_1), (a_\alpha)_2 + x_2((e_\alpha)_2 - (a_\alpha)_2)\}, \forall x \in L^I;
\]

\[
T_\alpha' = \Phi_\alpha^{-1} \circ T_\alpha \circ (\Phi_\alpha \times \Phi_\alpha).
\]

In Figure 2, the three smaller triangles are $J_\alpha$, $J_k$ and $J_\beta$. Assume that $J_\alpha^* \cap J_\beta^* = \emptyset$, for any $\alpha, \beta \in A$. Our aim is to construct a t-norm $T$ on $L^I$ such that $T|_{J_\alpha^* \times J_\beta^*} = T_\alpha'$, for all $\alpha \in A$.

Let arbitrarily $k \in A$ and define the sets $A_\subset = \{\alpha \mid \alpha \in A \text{ and } a_\alpha <_L a_k\}$ and $A_\supset = \{\alpha \mid \alpha \in A \text{ and } a_\alpha >_L a_k\}$. Assume furthermore that $T_\alpha([0,1], [0,1]) = [0,1]$, for all $\alpha \in A_\subset$, and $T_\alpha([0,1], [0,1]) = [0,0]$, for all $\alpha \in A_\supset$. For $T_k$ we do not impose any restriction, so $T_k([0,1], [0,1]) = [0,t]$ with $t \in [0,1]$. In [4] Theorem 4.2 it is shown that if $T_\alpha$ is continuous for all $\alpha \in A$ and if we want to construct a t-norm $T$ on $L^I$ which satisfies the residuation principle and for which $T|_{J_\alpha^* \times J_\beta^*} = T_\alpha'$ for all $\alpha \in A$, then there must exist a $k \in A$ such that the previously mentioned assumptions for $T_\alpha([0,1], [0,1])$, for all $\alpha \in A$, hold.

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\[\text{In [4] it is shown that if } a_\alpha \notin D \text{ or } e_\alpha \notin D, \text{ then there does not exist an increasing bijection } \Phi \text{ from } J_\alpha \text{ to } L^I \text{ such that } \Phi^{-1} \text{ is increasing. In this case the ordinal sum construction cannot be extended to } L^I.\]
Theorem 2.3 Let, for all $\alpha \in A$, $T_\alpha : [0, 1]^2 \to [0, 1]$ be the mapping defined by

$$T_\alpha(x_1, y_1) = (T_\alpha([x_1, x_1], [y_1, y_1]))_1, \forall (x_1, y_1) \in [0, 1]^2,$$

and let $T$ be the ordinal sum of $(\langle (a_\alpha)_1, (e_\alpha)_1, T_\alpha \rangle, \alpha \in A$. Define the mapping $T : (L^I)^2 \to L^I$ by, for all $x, y \in L^I$,

$$(T(x, y))_1 = T(x, y),$$

$$(T(x, y))_2 = \begin{cases} (T'_i[\max(x_1, (a_\alpha)_1), \min(x_2, (e_\alpha)_2)], \max(y_1, (a_\alpha)_1), \min(y_2, (e_\alpha)_2))]_2, \\
\quad \text{if } (x_2 \in ](a_\alpha)_2, (e_\alpha)_2] \text{ and } y_2 > (a_\alpha)_2 \text{ and } y_1 \leq (e_\alpha)_1 \text{ and } \alpha \in A_<) \text{ or } (y_2 \in ](a_\alpha)_2, (e_\alpha)_2] \text{ and } x_2 > (a_\alpha)_2 \text{ and } x_1 \leq (e_\alpha)_1 \text{ and } \alpha \in A_<) \\
\quad \text{or } (x_1 \in ](a_\alpha)_1, (e_\alpha)_1] \text{ and } y_2 > (a_\alpha)_2 \text{ and } y_1 \leq (e_\alpha)_1 \text{ and } \alpha \in A_>) \text{ or } (y_1 \in ](a_\alpha)_1, (e_\alpha)_1] \text{ and } x_2 > (a_\alpha)_2 \text{ and } x_1 \leq (e_\alpha)_1 \text{ and } \alpha \in A_>) \\
\quad \text{or } (x_2 > (a_\alpha)_2 \text{ and } x_1 \leq (e_\alpha)_1 \text{ and } y_2 > (a_\alpha)_2 \text{ and } y_1 \leq (e_\alpha)_1 \text{ and } \alpha = k), \\
\min(x_2, y_2), \text{ if the previous conditions do not hold} \\
\quad \text{and } x_2 \leq (a_k)_2 \text{ or } y_2 \leq (a_k)_2, \\
\min(x_2, y_1), \text{ if the previous conditions do not hold and } x_1 \leq y_1, \\
\min(y_2, x_1), \text{ else.} \end{cases}$$

Then $T$ is a t-norm on $L^I$ called the ordinal sum of the summands $\langle (a_\alpha, e_\alpha, T_\alpha) \rangle, \alpha \in A$, and we write

$$T = (((\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A_<} / \langle a_k, e_k, T_k \rangle / ((a_\alpha, e_\alpha, T_\alpha)_{\alpha \in A_>} \rangle.$$ 

In Figure 2 the construction of $(T(x_i, y_i))_2$ is shown for $(x_i, y_i) \in (L^I)^2$ where $i \in \{0, \ldots, 5\}$. The value of $(T(x_i, y_i))_2$ is calculated at the ending points of the arrows for each $i \in \{0, \ldots, 5\}$. In the figure, $k$ is defined as in the paragraph before Theorem 2.3 $\alpha \in A_<$ and $\beta \in A_>.$

In the following example we show that there exist different t-norms $T_1$ and $T_2$ on $([0, 1], \leq)$ such that the mapping $T_{T_1,T_2,t}$ defined in Example 2.1 is a t-norm on $L^I$.

Example 2.3 Let $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$ be t-norms on $([0, 1], \leq)$ such that $\hat{T}_1 \leq \hat{T}_2$. Let furthermore $t \in [0, 1]$. Define the t-norms $T_1$ and $T_2$ by

$$T_1 = ((0, t, \hat{T}_1), (t, 1, \hat{T}_3)),$$

$$T_2 = ((0, t, \hat{T}_2), (t, 1, \hat{T}_3)).$$

Then

$$T_2(x_1, y_1) > T_2(t, T_2(x_1, y_1)) = \min(t, T_2(x_1, y_1))$$

$$\iff T_2(x_1, y_1) > t$$

$$\implies \min(x_1, y_1) > t,$$

for all $x_1, y_1$ in $[0, 1]$. It can be easily verified that $T_1 \leq T_2$ and $T_1(x_1, y_1) = T_2(x_1, y_1)$, for all $x_1, y_1$ in $[0, 1]^2$. Clearly, if $\hat{T}_1 \neq \hat{T}_2$, then $T_1 \neq T_2$.

Define the mapping $T_{T_1,T_2,t}$ by $T_{T_1,T_2,t}(x, y) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(x_2, y_1))]$, for all $x, y$ in $L^I$. Then $T_{T_1,T_2,t}$ is a t-norm on $L^I$ (see Example 2.1).
Finally we need a metric on $L^I$. Well-known metrics include the Euclidean distance and the Hamming distance. In the two-dimensional space $\mathbb{R}^2$ they are defined as follows:

- the Euclidean distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ is given by
  \[ d^E(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \]
- the Hamming distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ is given by
  \[ d^H(x, y) = |x_1 - y_1| + |x_2 - y_2|. \]

If we restrict these distances to $L^I$ then we obtain the metric spaces $(L^I, d^E)$ and $(L^I, d^H)$. In these metric spaces, denote by $B(a; \varepsilon)$ the open ball with center $a$ and radius $\varepsilon$ defined as $B(a; \varepsilon) = \{ x \in L^I \mid d(x, a) < \varepsilon \}$. In the sequel, when we speak about continuity on $L^I$, we mean continuity w.r.t. one of the above mentioned metric spaces.

3 Characterization of t-norms which are meet-morphisms

Since $([0, 1], \leq)$ is a chain, any t-norm on the unit interval is a join- and a meet-morphism. Furthermore, it is well-known that continuous t-norms on $([0, 1], \leq)$ are sup- and infmorphisms. For t-norms on product lattices, the following result holds.

Theorem 3.1 \[3\] Consider two bounded lattices $L_1 = (L_1, \leq_{L_1})$ and $L_2 = (L_2, \leq_{L_2})$ and a t-norm $T$ on the product lattice $L_1 \times L_2 = (L_1 \times L_2, \leq)$, where $(x_1, x_2) \leq (y_1, y_2) \iff (x_1 \leq_{L_1} y_1$ and $x_2 \leq_{L_2} y_2)$, for all $(x_1, x_2), (y_1, y_2)$ in $L_1 \times L_2$. The t-norm $T$ is a join-morphism (resp. meet-morphism) if and only if there exist t-norms $T_1$ on $L_1$ and $T_2$ on $L_2$ which are join-morphisms (resp. meetmorphisms), such that for all $(x_1, x_2), (y_1, y_2)$ in $L_1 \times L_2$,

\[ T((x_1, x_2), (y_1, y_2)) = [T_1(x_1, y_1), T_2(x_2, y_2)]. \]

On $L^I$, the situation is more complicated. Not all t-norms on $L^I$ are join- and meetmorphisms. Consider the t-norm $T_{T^p}$ given by $T_{T^p}(x, y) = \min(x_1 y_2, x_2 y_2)$, for all $x, y$ in $L^I$. Then we have $T_{T^p}([0.2, 0.5], \sup([0.5, 0.5], [0.1])) = T_{T^p}([0.2, 0.5], [0.5, 1]) = [0.2, 0.5] \neq [0.1, 0.5] = \sup([0.1, 0.25], [0.5, 0.5]) = \sup(T_{T^p}([0.2, 0.5], [0.5, 0.5]), T_{T^p}([0.2, 0.5], [0, 1]))$. So $T_{T^p}$ is not a join-morphism. Similarly the t-norm $T_{T^p}$ is not a meet-morphism.

Gehrke et al. \[10\] used the following definition for a t-norm on $L^I$: a commutative, associative binary operation $T$ on $L^I$ is a t-norm if for all $x, y, z$ in $L^I$,

(G.1) $T(D, D) \subseteq D$,

(G.2) $T(x, \sup(y, z)) = \sup(T(x, y), T(x, z))$,

(G.3) $T(x, \inf(y, z)) = \inf(T(x, y), T(x, z))$,

(G.4) $T(1_{L^I}, x) = x$,

(G.5) $T([0, 1], x) = [0, x_2]$. 


They showed that such a t-norm is increasing, so their t-norms are a special case of the t-norms on \( L^I \) as defined in Definition 2.4.

Clearly, commutative, associative binary operations on \( L^I \) satisfying (G.1)–(G.5) are t-norms on \( L^I \) which are join- and meet-morphisms. The two additional conditions (G.1) and (G.5) ensure that these t-norms are t-representable, as is shown in the next theorem.

**Theorem 3.2** [10] For every commutative, associative binary operation \( T \) on \( L^I \) satisfying (G.1)–(G.5) there exists a t-norm \( T \) on \( ([0, 1], \leq) \) such that, for all \( x, y \) in \( L^I \),

\[
T(x, y) = [T(x_1, y_1), T(x_2, y_2)].
\]

We can extend this result as follows. First we need a lemma.

**Lemma 3.3** [5] Let \( T \) be a t-norm on \( L^I \) which is a join-morphism. Then there exists a t-norm \( T \) on \( ([0, 1], \leq) \) such that, for all \( x, y \) in \( L^I \),

\[
(T(x, y))_1 = T(x_1, y_1).
\]

**Theorem 3.4** For any t-norm \( T \) on \( L^I \) satisfying (G.2) and (G.5) there exist t-norms \( T_1 \) and \( T_2 \) on \( ([0, 1], \leq) \) such that, for all \( x, y \) in \( L^I \),

\[
T(x, y) = [T_1(x_1, y_1), T_2(x_2, y_2)].
\]

**Proof.** From Lemma 3.3 it follows that there exist a t-norm \( T_1 \) on \( ([0, 1], \leq) \) such that \( (T(x, y))_1 = T_1(x_1, y_1) \), for all \( x, y \) in \( L^I \). From (G.5) it follows that, for all \( x, y \) in \( L^I \),

\[
(T(x, y))_2 = (T([0, 1], T(x, y)))_2
= (T(T([0, 1], x), T([0, 1], y)))_2
= (T([0, x_2], [0, y_2]))_2.
\]

Hence \( (T(x, y))_2 \) is independent of \( x_1 \) and \( y_1 \), for all \( x, y \) in \( L^I \). Let now \( T_2(x_2, y_2) = (T([x_2, x_2], [y_2, y_2]))_2 \), for all \( x_2, y_2 \) in \( [0, 1] \). Similarly as in the proof of Lemma 3.3 given in [5] it is shown that \( T_2 \) is a t-norm on \( ([0, 1], \leq) \). \( \square \)

Clearly, (G.5) is a rather restrictive condition. We will show that if this condition is not imposed, then the class of t-norms on \( L^I \) satisfying the other conditions is much larger.

For continuous t-norms on \( L^I \) we have the following relationship between sup- and join-morphism, and between inf- and meet-morphisms.

**Theorem 3.5** Let \( T \) be a continuous t-norm on \( L^I \). Then

(i) \( T \) is a sup-morphism if and only if \( T \) is a join-morphism;

(ii) \( T \) is an inf-morphism if and only if \( T \) is a meet-morphism.

**Proof.** Let \( T \) be a continuous t-norm on \( L^I \). We prove the first statement, the second equivalence is proven in a similar way. Clearly, if \( T \) is a sup-morphism, then \( T \) is a join-morphism.
Assume conversely that $\mathcal{T}$ is a join-morphism. Let $x \in L^I$, $A$ be an arbitrary non-empty subset of $L^I$ and $a = \sup A$. Since $\mathcal{T}$ is increasing, we have that $\mathcal{T}(x,y) \leq_{L^I} \mathcal{T}(x,a)$, for all $y \in A$.

From Theorem 2.1 it follows that there exists a sequence $(y_n)_{n \in \mathbb{N}^*}$ in $A$ such that $(y_n)_1 > a_1 - \frac{1}{n}$, for all $n \in \mathbb{N}^*$. Let $y^* = \lim_{n \to +\infty} y_n$, then clearly $y^*_1 = a_1$ and $y^*_2 \leq a_2$. Similarly, there exists a sequence $(z_n)_{n \in \mathbb{N}^*}$ in $A$ such that $(z_n)_2 > a_2 - \frac{1}{n}$, for all $n \in \mathbb{N}^*$. Let $z^* = \lim_{n \to +\infty} z_n$, then $z^*_2 = a_2$ and $z^*_1 \leq a_1$. Since $\mathcal{T}$ is a join-morphism, $\mathcal{T}(x,a) = \sup(\mathcal{T}(x,y^*), \mathcal{T}(x,z^*)) = [\max((\mathcal{T}(x,y^*))_1, (\mathcal{T}(x,z^*))_1), \max((\mathcal{T}(x,y^*))_2, (\mathcal{T}(x,z^*))_2)]$.

Assume that $(\mathcal{T}(x,a))_1 = (\mathcal{T}(x,y^*))_1$ (the case $(\mathcal{T}(x,a))_1 = (\mathcal{T}(x,z^*))_1$ is similar). Since $\mathcal{T}$ is continuous, we have in particular that

\[
(\forall \varepsilon_1 > 0)(\exists N \in \mathbb{N}^*)(\forall n \in \mathbb{N}^*) \quad (n > N \implies |(\mathcal{T}(x,y_n))_1 - (\mathcal{T}(x,y^*))_1| + |(\mathcal{T}(x,y_n))_2 - (\mathcal{T}(x,y^*))_2| < \varepsilon_1).
\]

So, for any $\varepsilon_1 > 0$, there exists an $n \in \mathbb{N}^*$ such that $(\mathcal{T}(x,y^*))_1 - \varepsilon_1 < (\mathcal{T}(x,y_n))_1 \leq (\mathcal{T}(x,y^*))_1$. Hence, for any $\varepsilon_1 > 0$, there exists an element $y \in A$ such that $\mathcal{T}(x,y)_1 > (\mathcal{T}(x,a))_1 - \varepsilon_1$. Similarly, for any $\varepsilon_2 > 0$, there exists a $z \in A$ such that $\mathcal{T}(x,z)_2 > (\mathcal{T}(x,a))_2 - \varepsilon_2$. From Theorem 2.1 it follows that $\mathcal{T}(x,a) = \sup_{y \in A} \mathcal{T}(x,y)$. $\Box$

In the following theorem the t-norms on $L^I$ which satisfy the residuation principle and an additional border condition are characterized in terms of the class of t-norms $\mathcal{T}_{T_1,T_2,t}$ given in Example 2.1.

**Theorem 3.6** [2] Let $\mathcal{T} : (L^I)^2 \to L^I$ be a t-norm such that, for all $x \in D$, $y_2 \in [0,1]$, $(\mathcal{T}(x,[y_2,y_2]))_2 = (\mathcal{T}(x,[0,y_2]))_2$. Then $\mathcal{T}$ satisfies the residuation principle if and only if there exist two left-continuous t-norms $T_1$ and $T_2$ on $([0,1], \leq)$ and a real number $t \in [0,1]$ such that, for all $x,y \in L^I$,

\[
\mathcal{T}(x,y) = [T_1(x_1,y_1), \max(T_2(t,T_2(x_2,y_2)),T_2(x_1,y_2),T_2(y_1,x_2))],
\]

i.e. $\mathcal{T} = \mathcal{T}_{T_1,T_2,t}$, and, for all $x_1,y_1$ in $[0,1]$,

\[
\begin{cases}
T_1(x_1,y_1) = T_2(x_1,y_1), & \text{if } T_2(x_1,y_1) > T_2(t,T_2(x_1,y_1)), \\
T_1(x_1,y_1) \leq T_2(x_1,y_1), & \text{else}.
\end{cases}
\]

We extend Theorem 3.6 to t-norms on $L^I$ which are join-morphisms. The proof of the following theorem is analogous to the proof of Theorem 3.6 given in [2].

**Theorem 3.7** Let $\mathcal{T} : (L^I)^2 \to L^I$ be a t-norm such that, for all $x \in D$, $y_2 \in [0,1]$, $(\mathcal{T}(x,[y_2,y_2]))_2 = (\mathcal{T}(x,[0,y_2]))_2$. Then $\mathcal{T}$ is a join-morphism if and only if there exist two t-norms $T_1$ and $T_2$ on $([0,1], \leq)$ and a real number $t \in [0,1]$ such that, for all $x,y \in L^I$,

\[
\mathcal{T}(x,y) = [T_1(x_1,y_1), \max(T_2(t,T_2(x_2,y_2)),T_2(x_1,y_2),T_2(y_1,x_2))],
\]

i.e. $\mathcal{T} = \mathcal{T}_{T_1,T_2,t}$, and, for all $x_1,y_1$ in $[0,1]$,

\[
\begin{cases}
T_1(x_1,y_1) = T_2(x_1,y_1), & \text{if } T_2(x_1,y_1) > T_2(t,T_2(x_1,y_1)), \\
T_1(x_1,y_1) \leq T_2(x_1,y_1), & \text{else}.
\end{cases}
\]

9
Now we characterize the t-norms on \( L^I \) belonging to the class \( T_{T_1,T_2,t} \) which are meet-morphisms. First we need some lemmas.

**Lemma 3.8** Assume that \( T_{T_1,T_2,t} \) is a meet-morphism. Then \( T_2(t,y_1) = \min(t,y_1) \), for all \( y_1 \in [0,1] \).

**Proof.** Let arbitrarily \( y_1 \in [0,1] \). Then

\[
T_{T_1,T_2,t}([0,1], \inf([y_1,y_1],[0,1])) = T_{T_1,T_2,t}([0,1],[0,y_1]) = [0, T_2(t,T_2(1,y_1))] = [0, T_2(t,y_1)].
\]

On the other hand,

\[
T_{T_1,T_2,t}([0,1], \inf([y_1,y_1],[0,1])) = \inf(T_{T_1,T_2,t}([0,1],[y_1,y_1]), T_{T_1,T_2,t}([0,1],[0,1])) = \inf([0,\max(T_2(t,y_1),y_1),[0,t]) = \inf([0,y_1],[0,t]) = [0,\min(y_1,t)].
\]

Hence \( T_2(t,y_1) = \min(t,y_1) \), for all \( y_1 \in [0,1] \). \( \square \)

**Corollary 3.9** Assume that \( T_{T_1,T_2,t} \) is a meet-morphism. Then there exists two t-norms \( \hat{T}_1 \) and \( \hat{T}_2 \) on \(([0,1],\leq)\) such that

\[
T_2 = (\langle 0,t,\hat{T}_1 \rangle, \langle t,1,\hat{T}_2 \rangle).
\]

**Proof.** Define, for all \( x,y \) in \([0,1]\),

\[
\hat{T}_1(x,y) = \frac{T_2(tx,ty)}{t}, \quad \hat{T}_2(x,y) = \frac{T_2(t+(1-t)x, t+(1-t)y) - t}{1-t}.
\]

Then it is easy to see that \( \hat{T}_1 \) is commutative, associative and increasing. Since from Lemma 3.8 it follows that \( T_2(t,y) = \min(t,y) \), for all \( y \in [0,1] \), we obtain that \( \hat{T}_1(1,y) = y \), for all \( y \in [0,1] \). So \( \hat{T}_1 \) is a t-norm. Similarly, we obtain that \( \hat{T}_2 \) is a t-norm on \( ([0,1],\leq) \).

Let arbitrarily \( x,y \) in \([0,1]\) such that \( x < t < y \) (the case \( y < t < x \) is similar). Then we obtain that \( x = \min(t,x) = T_2(t,x) \leq T_2(x,y) \leq T_2(1,x) = x \), so \( T_2(x,y) = \min(x,y) \). It now easily follows that \( T_2 \) is equal to the ordinal sum of \( \langle 0,t,\hat{T}_1 \rangle \) and \( \langle t,1,\hat{T}_2 \rangle \). \( \square \)

**Lemma 3.10** Assume that \( T_{T_1,T_2,t} \) is a meet-morphism. Then the t-norm \( \hat{T}_2 \) in the representation of \( T_2 \) given in Corollary 3.9 is equal to the minimum.
Proof. Let arbitrarily \( x_1, z_1 \) in \([t, 1]\). From Lemma \([3,8]\) it follows that \( T_2(t, z_1) = \min(t, z_1) = t \). Furthermore, from Corollary \([3,9]\) it follows that \( T_2(x_1, z_1) \geq t \). So, we obtain

\[
T_{T_1,T_2,t}([x_1, 1], \inf([0, 1], [z_1, z_1])) = T_{T_1,T_2,t}([x_1, 1], [0, z_1]) = [0, \max(T_2(t, z_1), T_2(x_1, z_1))] = [0, \max(t, T_2(x_1, z_1))] = [0, T_2(x_1, z_1)]
\]

and

\[
T_{T_1,T_2,t}([x_1, 1], \inf([0, 1], [z_1, z_1])) = \inf(T_{T_1,T_2,t}([x_1, 1], [0, 1]), T_{T_1,T_2,t}([x_1, 1], [z_1, z_1])) = \inf([0, \max(t, x_1)], [T_1(x_1, z_1), \max(T_2(t, z_1), T_2(x_1, z_1))]) = \inf([0, x_1], [T_1(x_1, z_1), z_1]) = [0, \min(x_1, z_1)].
\]

So \( T_2(x_1, z_1) = \min(x_1, z_1) \). From \([3]\) it easily follows that \( \hat{T}_2 = \min \).

\[
\]

Corollary 3.11 Assume that \( T_{T_1,T_2,t} \) is a meet-morphism. Then there exists a t-norm \( \hat{T}_1 \) on \(([0, 1], \leq)\) such that

\[ T_2 = (\{0, t, \hat{T}_1\}, (t, 1, \min)) \]

Lemma 3.12 Assume that there exists a t-norm \( \hat{T}_1 \) on \(([0, 1], \leq)\) such that \( T_2 = (\{0, t, \hat{T}_1\}, (t, 1, \min)) \), then \( T_{T_1,T_2,t} \) is a meet-morphism.

Proof. Let arbitrarily \( x, y, z \) in \( L^I \). If \( y \leq_L z \) (the case \( y \geq_L z \) is similar), then

\( T_{T_1,T_2,t}(x, \inf(y, z)) = T_{T_1,T_2,t}(x, y) = \inf(T_{T_1,T_2,t}(x, y), T_{T_1,T_2,t}(x, z)) \). So, let \( y_1 < z_1 \) and \( y_2 > z_2 \) (the case \( y_1 > z_1 \) and \( y_2 < z_2 \) is similar). Then we have the following cases:

- \( \max(x_1, y_1, z_1) \leq t \):
  - From the fact that \( T_2 \leq \min \) it follows that \( T_2(x_1, z_2) \leq t \) and \( T_2(x_2, y_1) \leq t \), so \( T_2(x_1, z_2) \leq \min(t, T_2(x_2, z_2)) = T_2(t, T_2(x_2, z_2)) \). Since \( T_2(x_2, y_1) \leq T_2(x_2, z_1) \leq T_2(x_2, z_2) \), we obtain similarly that \( T_2(x_2, y_1) \leq T_2(t, T_2(x_2, z_2)) \). Thus,

\[
T_{T_1,T_2,t}(x, \inf(y, z)) = T_{T_1,T_2,t}(x, [y_1, z_2]) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_1, z_2), T_2(x_2, y_1))] = [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))].
\]

On the other hand, we obtain similarly that

\[
\inf(T_{T_1,T_2,t}(x, y), T_{T_1,T_2,t}(x, z)) = \inf([T_1(x_1, y_1), T_2(t, T_2(x_2, y_2))], [T_1(x_1, z_1), T_2(t, T_2(x_2, z_2))]) = [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))],
\]

using the fact that \( T_2 \) is increasing, \( y_1 < z_1 \) and \( y_2 > z_2 \).
• $\max(x_1, y_1) \leq t < z_1$:

Similarly as in the previous case, we have that

$$\mathcal{T}_{T_1, T_2, t}(x, \inf(y, z)) = [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))]$$

and

$$\inf(\mathcal{T}_{T_1, T_2, t}(x, y), \mathcal{T}_{T_1, T_2, t}(x, z)) = \inf([T_1(x_1, y_1), T_2(t, T_2(x_2, y_2))], [T_1(x_1, z_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_2, z_1))])$$

$$= [T_1(x_1, y_1), \min(T_2(t, T_2(x_2, y_2)), \max(\min(t, T_2(x_2, z_2)), T_2(x_2, z_1))]].$$

We have two cases:

1. $x_2 \leq t$: in this case, we have that $T_2(x_2, z_1) = \min(x_2, z_1) = x_2 \leq t$, so $T_2(x_2, z_1) \leq \min(t, T_2(x_2, z_2)) = T_2(t, T_2(x_2, z_2))$. Hence

$$\inf(\mathcal{T}_{T_1, T_2, t}(x, y), \mathcal{T}_{T_1, T_2, t}(x, z)) = [T_1(x_1, y_1), \min(T_2(t, T_2(x_2, y_2)), T_2(t, T_2(x_2, z_2)))]$$

$$= [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))].$$

2. $x_2 > t$: in this case, $T_2(x_2, z_1) = \min(x_2, z_1) > t$, so $T_2(x_2, y_2) \geq T_2(x_2, z_2) \geq T_2(x_2, z_1) > t$. Thus,

$$\inf(\mathcal{T}_{T_1, T_2, t}(x, y), \mathcal{T}_{T_1, T_2, t}(x, z)) = [T_1(x_1, y_1), \min(\min(t, T_2(x_2, y_2)), T_2(x_2, z_1))]$$

$$= [T_1(x_1, y_1), t]$$

and

$$\mathcal{T}_{T_1, T_2, t}(x, \inf(y, z)) = [T_1(x_1, y_1), \min(t, T_2(x_2, z_2))] = [T_1(x_1, y_1), t].$$

• $x_1 \leq t < y_1 (< z_1)$:

We have that $T_2(x_1, z_2) \leq x_1 \leq t$, so $T_2(x_1, z_2) \leq \min(t, T_2(x_2, z_2)) = T_2(t, T_2(x_2, z_2))$. We obtain

$$\mathcal{T}_{T_1, T_2, t}(x, \inf(y, z)) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_2, y_1))]$$

and similarly

$$\inf(\mathcal{T}_{T_1, T_2, t}(x, y), \mathcal{T}_{T_1, T_2, t}(x, z)) = \inf([T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_2, y_1))], [T_1(x_1, z_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_2, z_1))]).$$

We have two cases:

1. $x_2 \leq t$: in this case, we have that $T_2(x_2, y_1) \leq t$, so, using the fact that $y_1 < z_1 \leq z_2$, $T_2(x_2, y_1) \leq \min(t, T_2(x_2, z_2)) = T_2(t, T_2(x_2, z_2))$. Thus,

$$\mathcal{T}_{T_1, T_2, t}(x, \inf(y, z)) = [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))].$$

Similarly, we obtain that $\inf(\mathcal{T}_{T_1, T_2, t}(x, y), \mathcal{T}_{T_1, T_2, t}(x, z)) = [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))].$
2. \( x_2 > t \): from the representation of \( T_2 \) it follows that \( T_2(x_2, y_2) \geq T_2(x_2, z_2) \geq T_2(x_2, z_1) \geq t \). So, using the fact that \( T_2(t, a) = \min(t, a) \) for all \( a \in [0, 1] \), we obtain
\[
T_{T_1, T_2, t}(x, \inf(y, z)) = [T_1(x_1, y_1), \max(t, T_2(x_2, y_1))]
\]
and
\[
\inf(T_{T_1, T_2, t}(x, y), T_{T_1, T_2, t}(x, z)) = [T_1(x_1, y_1), \min(\max(t, T_2(x_2, y_1)), \max(t, T_2(x_2, z_1)))]
= [T_1(x_1, y_1), \max(t, T_2(x_2, y_1))].
\]

• \((y_1 < z_1) \leq t < x_1\):
Similarly as in the previous case, we obtain that
\[
T_{T_1, T_2, t}(x, \inf(y, z)) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_2, z_2))]
\]
and
\[
\inf(T_{T_1, T_2, t}(x, y), T_{T_1, T_2, t}(x, z)) = \inf([T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_2, y_2))],
[T_1(x_1, z_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_2, z_2))]).
\]

We have two cases:

1. \( y_2 \leq t \): we obtain that \( T_2(x_1, z_2) \leq T_2(x_1, y_2) \leq t \), so \( T_2(x_1, y_2) \leq \min(t, T_2(x_2, y_2)) = T_2(t, T_2(x_2, y_2)) \) and similarly for \( T_2(x_2, z_2) \). Thus
\[
T_{T_1, T_2, t}(x, \inf(y, z)) = [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))]
\]
and
\[
\inf(T_{T_1, T_2, t}(x, y), T_{T_1, T_2, t}(x, z)) = \inf([T_1(x_1, y_1), T_2(t, T_2(x_2, y_2))], [T_1(x_1, z_1), T_2(t, T_2(x_2, z_2))])
= [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))].
\]

2. \( y_2 > t \): we have that \( T_2(x_1, y_2) \geq t \geq \min(t, T_2(x_2, z_2)) \) and \( T_2(x_1, y_2) \geq T_2(x_1, z_2) \), so
\[
\inf(T_{T_1, T_2, t}(x, y), T_{T_1, T_2, t}(x, z)) = [T_1(x_1, y_1), \min(T_2(x_1, y_2), \max(T_2(t, T_2(x_2, z_2)), T_2(x_1, z_2)))]
= [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_1, z_2))]
= T_{T_1, T_2, t}(x, \inf(y, z)).
\]

• \( y_1 \leq t < \min(x_1, z_1) \):
We have that \( T_2(x_2, y_1) \leq y_1 \leq t \leq T_2(x_1, z_2) \leq T_2(x_1, y_2) \), so
\[
T_{T_1, T_2, t}(x, \inf(y, z)) = [T_1(x_1, y_1), \max(\min(t, T_2(x_2, z_2)), T_2(x_1, z_2))]
= [T_1(x_1, y_1), T_2(x_1, z_2)].
\]
Similarly,

\[
\inf(\mathcal{T}_{T_1, T_2, t}(x, y), \mathcal{T}_{T_1, T_2, t}(x, z)) = \inf([T_1(x_1, y_1), \max(T_2(x_1, y_2), T_2(x_2, y_1))], [T_1(x_1, z_1), \max(T_2(x_1, z_2), T_2(x_2, z_1))]) = [T_1(x_1, y_1), \min(T_2(x_1, y_2), \max(T_2(x_1, z_2), T_2(x_2, z_1))]).
\]

We have two cases:

1. \(x_1 < \min(x_2, z_1)\): in this case, we have that \(T_2(x_1, z_2) = \min(x_1, z_2) = x_1 < \min(x_2, z_1) = T_2(x_2, z_1)\) (using Corollary \[3.11\]), so
   \[
   \min(T_2(x_1, y_2), \max(T_2(x_1, z_2), T_2(x_2, z_1))) = \min(T_2(x_1, y_2), T_2(x_2, z_1)) = \min(x_1, y_2, x_2, z_1) = x_1 = \min(x_1, z_2) = T_2(x_1, z_2).
   \]

2. \(x_1 \geq \min(x_2, z_1)\): since \(z_2 \geq z_1 \geq \min(x_2, z_1)\), we have that \(T_2(x_1, z_2) = \min(x_1, z_2) \geq \min(x_2, z_1) = T_2(x_2, z_1)\), so
   \[
   \min(T_2(x_1, y_2), \max(T_2(x_1, z_2), T_2(x_2, z_1))) = \min(T_2(x_1, y_2), T_2(x_1, z_2)) = T_2(x_1, z_2),
   \]
   since \(y_2 > z_2\).

- \(t \leq \min(x_1, y_1, z_1)\):

From Lemma \[3.8\] and Corollary \[3.11\] it follows that

\[
\mathcal{T}_{T_1, T_2, t}(x, \inf(y, z)) = \mathcal{T}_{T_1, T_2, t}(x, [y_1, z_2]) = [T_1(x_1, y_1), \max(\min(t, T_2(x_2, z_2)), \min(x_1, z_2), \min(x_2, y_1))] = [T_1(x_1, y_1), \max(\min(x_1, z_2), \min(x_2, y_1))].
\]

On the other hand, we obtain similarly that

\[
\inf(\mathcal{T}_{T_1, T_2, t}(x, y), \mathcal{T}_{T_1, T_2, t}(x, z)) = \inf([T_1(x_1, y_1), \max(x_1, y_2), \min(x_2, y_1)], [T_1(x_1, z_1), \max(\min(x_1, z_2), \min(x_2, z_1))]).
\]

Clearly, it holds that \((\mathcal{T}_{T_1, T_2, t}(x, \inf(y, z))_1 = T_1(x_1, y_1) = \min(T_1(x_1, y_1), T_1(x_1, z_1)) = (\inf(\mathcal{T}_{T_1, T_2, t}(x, y), \mathcal{T}_{T_1, T_2, t}(x, z))_1)\). For the second projection, we have two cases:

1. \(x_1 < \min(x_2, z_1)\): in this case, we have that \(\min(x_1, z_2) = x_1 < \min(x_2, z_1) \leq z_2 < y_2\). So, \((\mathcal{T}_{T_1, T_2, t}(x, \inf(y, z))_2 = \max(x_1, \min(x_2, y_1))\). On the other hand
   \[
   (\inf(\mathcal{T}_{T_1, T_2, t}(x, y), \mathcal{T}_{T_1, T_2, t}(x, z))_2 = \min(\max(x_1, \min(x_2, y_1)), \min(x_2, z_1)) = \max(x_1, \min(x_2, y_1)) = (\mathcal{T}_{T_1, T_2, t}(x, \inf(y, z))_2
   \]
   using the fact that \(y_1 < z_1\) and \(x_1 < \min(x_2, z_1)\).
2. \( x_1 \geq \min(x_2, z_1) \): in this case, we have that \( x_1 = x_2 \) or \( x_1 \geq z_1 \), so \( \min(x_1, z_2) \geq \min(x_2, z_1) \). If \( x_1 = x_2 \), then \( (T_{T_2,T_2,t}(x, \min(y, z)))_2 = \min(x_1, z_2) \), because \( z_2 \geq z_1 > y_1 \). On the other hand, \( (\inf(T_{T_1,T_2,t}(x, y), T_{T_1,T_2,t}(x, z)))_2 = \min(\min(x_1, y_2), \min(x_1, z_2)) = \min(x_1, z_2) \).

If \( x_1 \geq z_1 \), then \( (T_{T_1,T_2,t}(x, \min(y, z)))_2 = \max(\min(x_1, z_2), y_1) = \min(x_1, z_2) \), because \( y_1 < z_1 \leq x_1 \leq x_2 \). On the other hand, \( (\inf(T_{T_1,T_2,t}(x, y), T_{T_1,T_2,t}(x, z)))_2 = \min(\max(\min(x_1, y_2), y_1), \min(x_1, z_2)) = \min(x_1, z_2) \), using the fact that \( z_2 < y_2 \). So again \( (\inf(T_{T_1,T_2,t}(x, y), T_{T_1,T_2,t}(x, z)))_2 = (T_{T_1,T_2,t}(x, \min(y, z)))_2 \).

Now we obtain the main result.

**Theorem 3.13** For any \( t \)-norms \( T_1 \) and \( T_2 \) on \( ([0,1], \leq) \) and \( t \in [0,1] \), \( T_{T_1,T_2,t} \) is a meet-morphism if and only if there exists a \( t \)-norm \( T_1 \) on \( ([0,1], \leq) \) such that

\[
T_2 = (\langle 0, T_1 \rangle, (t, 1, \min))
\]

**Proof.** This follows immediately from Corollary 3.11 and Lemma 3.12.

If we assume that \( T_1 = T_2 \), then we do not only obtain that \( T_1 \) is the ordinal sum of two \( t \)-norms on \( ([0,1], \leq) \), but we can also write the \( t \)-norm \( T_{T_1,T_2,t} \) as an ordinal sum of two \( t \)-norms on \( L^t \). This is shown in the next theorem.

**Theorem 3.14** For any \( t \)-norm \( T \) on \( ([0,1], \leq) \) and \( t \in [0,1] \), \( T_{T,t} \) is a meet-morphism if and only if there exists a \( t \)-norm \( T_1 \) on \( ([0,1], \leq) \) such that

\[
T_{T,t} = (\varnothing / \langle 0_{L^t}, [t, t], T_{T_1,T_1} \rangle / \langle [t, t], 1_{L^t}, T_{\min} \rangle)
\]

where, for all \( x, y \) in \( L^t \),

\[
T_{T_1,T_1}(x, y) = \left[ T_1(x_1, x_1), T_1(x_1, y_1) \right],
\]

\[
T_{\min}(x, y) = \left[ \min(x_1, y_1), \max(\min(x_1, y_2), \min(x_1, y_1)) \right]
\]

**Proof.** Assume first that \( T_{T,t} \) is a meet-morphism. From Theorem 3.13 it follows that there exists a \( t \)-norm \( T_1 \) on \( ([0,1], \leq) \) such that \( T = (\langle 0, T_1 \rangle, (t, 1, \min)) \).

Let \( \phi : [0, t] \to [0, 1] : x \mapsto \frac{x}{t} \) and \( T'_1 = \phi^{-1} \circ T_1 \circ (\phi \times \phi) \). Define for all \( x, y \) in \( L^t \),

\[
\Phi_1(x) = \left[ \phi(x_1), \phi(x_2) \right],
\]

\[
\Phi_2(x) = \left[ \frac{x_1 - t}{1 - t}, \frac{x_2 - t}{1 - t} \right],
\]

\[
T'_{T_1,T_1} = \Phi_1^{-1} \circ T_{T_1,T_1} \circ (\Phi_1 \times \Phi_1),
\]

\[
T'_{\min} = \Phi_2^{-1} \circ T_{\min} \circ (\Phi_2 \times \Phi_2).
\]

Note that \( T'_{\min} \) defined by the formula above is a transformation of \( T'_{\min} \) and not a member of the class of \( t \)-norms \( T_{T,t} \) given in Example 2.1. Then, for all \( x, y, x', y' \) in \( L^t \) such that

\[
x \leq_{L^t} [t, t], y \leq_{L^t} [t, t], x' \geq_{L^t} [t, t] \text{ and } y' \geq_{L^t} [t, t],
\]

\[
T'_{T_1,T_1}(x, y) = \left[ T'_1(x_1, y_1), T'_1(x_2, y_2) \right],
\]

\[
T'_{\min}(x', y') = \left[ \min(x'_1, y'_1), \max(\min(x'_1, y'_2), \min(x'_1, y'_1)) \right]
\]

We consider the following cases:

15
1. \( \max(x_2, y_2) \leq t \): using Lemma 3.8 we obtain
\[
\mathcal{T}_{T,t}(x, y) = [T(x_1, y_1), \max(\min(t, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))]
= [T(x_1, y_1), \max(T(x_2, y_2), T(x_1, y_2), T(x_2, y_1))]
= [\hat{T}_1(x_1, y_1), \hat{T}_1(x_2, y_2)].
\]

2. \( \max(x_2, y_1) \leq t < y_2 \) (the case \( \max(y_2, x_1) \leq t < x_2 \) is similar): we obtain in a completely similar way that \( \mathcal{T}_{T,t}(x, y) = [\hat{T}_1(x_1, y_1), \min(x_2, y_2)] = [\hat{T}_1(x_1, y_1), x_2] = [\hat{T}_1(x_1, y_1), \hat{T}_1(x_2, t)]. \)

3. \( \max(x_1, y_1) \leq t < \min(x_2, y_2) \): we obtain that \( T(t, T(x_2, y_2)) = \min(t, x_2, y_2) = t, T(x_1, y_2) \leq x_1 \leq t \) and \( T(x_2, y_1) \leq y_1 \leq t \). So \( \mathcal{T}_{T,t}(x, y) = [T(x_1, y_1), t] = [\hat{T}_1(x_1, y_1), \hat{T}_1(t, t)]. \)

4. \( x_2 \leq t < y_1 \) (the case \( y_2 \leq t < x_1 \) is similar): we obtain that \( T(t, T(x_2, y_2)) = \min(t, x_2, y_2) = x_2, T(x_1, y_2) = \min(x_1, y_2) = x_1 \) and \( T(x_2, y_1) = \min(x_2, y_1) = x_2 \). So \( \mathcal{T}_{T,t}(x, y) = [T(x_1, y_1), x_2] = [\min(x_1, y_1), \min(x_2, y_1)] = [\min(x_1, y_1), \min(x_2, y_2)]. \)

5. \( x_1 \leq t < \min(x_2, y_1) \) (the case \( y_1 \leq t < \min(y_2, x_1) \) is similar): we obtain that \( T(t, T(x_2, y_2)) = \min(t, x_2, y_2) = t, T(x_1, y_2) = \min(x_1, y_2) = x_1 \) and \( T(x_2, y_1) = \min(x_2, y_1) > t \). So \( \mathcal{T}_{T,t}(x, y) = [T(x_1, y_1), \min(x_2, y_1)] = [\min(x_1, y_1), \max(\min(t, y_2), \min(x_2, y_1))]. \)

6. \( t < \min(x_1, y_1) \): we obtain that \( T(t, T(x_2, y_2)) = \min(t, x_2, y_2) = t, \) so \( \mathcal{T}_{T,t}(x, y) = [\min(x_1, y_1), \max(\min(x_1, y_2), \min(x_2, y_1))]. \)

We see that
\[
(\mathcal{T}_{T,t}(x, y))_1 = T(x_1, y_1) = \begin{cases} 
\hat{T}_1(x_1, y_1), & \text{if } (x_1, y_1) \in [0, t]^2, \\
\min(x_1, y_1), & \text{else}.
\end{cases}
\]

So, the first projection of \( \mathcal{T}_{T,t} \) is determined by the ordinal sum of \( \langle 0, t, \hat{T}_1 \rangle \) and \( \langle t, 1, \min \rangle \). The second projection of \( \mathcal{T}_{T,t} \) is given by
\[
(\mathcal{T}_{T,t}(x, y))_2 = \begin{cases} 
(\mathcal{T}_{T_1,T_1}([x_1, \min(x_2, t)], [y_1, \min(y_2, t)]))_2, \\
\text{if } x_2 > 0 \text{ and } x_1 \leq t \text{ and } y_2 > 0 \text{ and } y_1 \leq t,
(\mathcal{T}_{\min}([\max(x_1, t), x_2], [\max(y_1, t), y_2]))_2, \\
\text{if } (x_1 \in [t, 1] \text{ and } y_2 > t \text{ and } y_1 \leq 1) \text{ or } (y_1 \in [t, 1] \text{ and } x_2 > t \text{ and } x_1 \leq 1),
\min(x_2, y_2), \text{ if the previous conditions do not hold and } x_2 \leq 0 \text{ or } y_2 \leq 0,
\min(x_2, y_1), \text{ if the previous conditions do not hold and } x_1 \leq y_1,
\min(y_2, x_1), \text{ else}.
\end{cases}
\]

This corresponds to the formula in Theorem 2.3, in which \( A = \{1, 2\}, a_1 = 0_{\mathcal{L}_t}, e_1 = a_2 = [t, t], e_2 = 1_{\mathcal{L}_t}, k = 1, A_\prec = \emptyset \) and \( A_\sim = \{2\} \). Hence \( \mathcal{T}_{T,t} \) is the ordinal sum of the summands \( \langle 0_{\mathcal{L}_t}, [t, t], \hat{T}_1, \hat{T}_1 \rangle \) and \( \langle [t, t], 1_{\mathcal{L}_t}, 1_{\mathcal{L}_t}, 1_{\mathcal{L}_t}, \min \rangle \), with \( k = 1 \).
Conversely, assume that $\mathcal{T}_{T,t}$ is the ordinal sum of the summands $\langle 0, t,\hat{T}_1,\hat{T}_1 \rangle$ and $\langle [t, t], 1,\hat{T}_1,\hat{T}_1 \rangle$, with $k = 1$. Then from Theorem 2.3 it follows that $T$ is the ordinal sum of $\langle 0, t,\hat{T}_1 \rangle$ and $\langle [t, t], 1,\min \rangle$. Using Theorem 3.13 we obtain that $\mathcal{T}_{T,t}$ is a meet-morphism. □

**Corollary 3.15** Let $T$ be a t-norm on $([0, 1], \leq)$.

- If $t = 0$, then $\mathcal{T}_{T,0}$ is a meet-morphism if and only if $\mathcal{T}_{T,0} = \mathcal{T}_{\min}$.
- If $t = 1$, then $\mathcal{T}_{T,1} = \mathcal{T}_{T,T}$ is a meet-morphism for any $T$.

By combining Theorems 3.6 and 3.13 we obtain the following result.

**Theorem 3.16** Let $\mathcal{T} : (L^1)^2 \to L^1$ be a t-norm such that, for all $x \in D$, $y_2 \in [0, 1]$, $(\mathcal{T}(x, [y_2, y_2]))_2 = (\mathcal{T}(x, [0, y_2]))_2$. Then $\mathcal{T}$ is a join-morphism and a meet-morphism if and only if there exist two t-norms $T_1$ and $T_2$ on $([0, 1], \leq)$ and a real number $t \in [0, 1]$ such that, for all $x, y \in L^1$,

$$\mathcal{T}(x, y) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(y_1, x_2))],$$

$T_2$ is the ordinal sum $\langle [0, t,\hat{T}_1], \langle [t, 1,\min] \rangle$, where $\hat{T}_1$ is a t-norm on $([0, 1], \leq)$, and, for all $x_1, y_1$ in $[0, 1]$,

$$T_1(x_1, y_1) = T_2(x_1, y_1), \text{ if } T_2(x_1, y_1) > t.$$

**4 Conclusion**

In this paper we investigated t-norms in interval-valued fuzzy set theory which are meet-morphisms. First we showed that for continuous t-norms the notions of sup- and join-morphism, respectively the notions of inf- and meet-morphism, collapse. We considered a general class of t-norms (given in [7]) and investigated under which conditions t-norms belonging to this class are meet-morphisms. We also showed that there exist non-trivial examples of t-norms in this class, i.e. t-norms which belong to this class but not to the class investigated in [5, 18]. Finally we gave a characterization of the t-norms which are both a join- and a meet-morphism and which satisfy an additional border condition.

**References**


