Phase diagram of chirally imbalanced QCD matter

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We compute the QCD phase diagram in the plane of the chiral chemical potential and temperature using the linear sigma model coupled to quarks and to the Polyakov loop. The chiral chemical potential accounts for effects of imbalanced chirality due to QCD sphaleron transitions which may emerge in heavy-ion collisions. We found three effects caused by the chiral chemical potential: the imbalanced chirality (i) tightens the link between deconfinement and chiral phase transitions; (ii) lowers the common critical temperature; (iii) strengthens the order of the phase transition by converting the crossover into the strong first order phase transition passing via the second order end point. Since the fermionic determinant with the chiral chemical potential has no sign problem, the chirally imbalanced QCD matter can be studied in numerical lattice simulations.

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\section{I. INTRODUCTION}

It is well known that the QCD vacuum has a nontrivial topological structure due to the presence of certain gluon configurations, instantons, which are characterized by an integer-valued topological winding number \cite{1}. For a long time, experimental evidence for the existence of the topological gluon configurations could only be found indirectly, in certain features of the meson spectrum \cite{2}.

Recently, it was noticed that a potentially observable direct signature of the topologically nontrivial gluon configurations can emerge in noncentral collisions of heavy ions \cite{3,4}. Such collisions create hot expanding fireballs of the quark-gluon plasma in the background of a strong magnetic field. Topologically nontrivial sphaleron transitions \cite{5,6} can induce—acting via the axial anomaly—a nonvanishing chiral density of quarks in the plasma fireballs. This chirally imbalanced matter is characterized by different densities of right- and left-handed quarks. If such medium is placed in an external magnetic field, then an electric current of quarks should emerge along the field’s axis. The unusual generation of the electric current in the magnetic field background is the essence of the so-called “chiral magnetic effect” \cite{3,4}. Signatures of this phenomenon were searched for in heavy-ion experiments at the Brookhaven National Laboratory’s Relativistic Heavy Ion Collider (RHIC) \cite{7} and they may probably be observed in the heavy-ion collisions at the Large Hadron Collider (LHC) at CERN.

The chiral magnetic effect is realized in the chirally imbalanced medium in the strong magnetic field background. In order for the chiral magnetic effect to be realized, the system of quarks and gluons should be in the deconfined phase, and the chiral symmetry should simultaneously be restored. Both conditions—deconfinement and chiral symmetry restoration—are realized at high temperature. The deconfinement requirement is essential because the system should be able to generate the electric current of individual quarks. Moreover, in the chirally broken (low temperature) phase, the chiral imbalance should quickly be washed out due to the presence of the chiral condensate which facilitates transitions between left-handed and right-handed quarks. Therefore, it becomes interesting to investigate the influence of the background magnetic field and the effect of the chiral imbalance on the finite-temperature transition between hadron and quark-gluon plasma phases.

The effect of the strong magnetic field on the QCD phase transition has been studied both analytically \cite{8–11} and numerically \cite{12}. All these studies have found that the magnetic field background increases the transition temperature and makes the phase transition stronger. A third effect was possibly found in Ref. \cite{10}: the magnetic field may split the deconfining and chiral phase transitions, thus leading to emergence of the new, chirally broken deconfining phase. (The splitting can be small \cite{11}, however.) In the low-temperature and strong-magnetic-field corner of the QCD phase diagram, a new electromagnetically superconducting phase may emerge \cite{13}.

The topologically induced changes in chirality can be modeled with the help of the chiral chemical potential $\mu_5$ which creates a difference between the right- and left-handed particles \cite{4}. This potential can be related to the $\theta$ angle of strong interactions as follows \cite{4}:

$$\mu_5 = \frac{\partial}{\partial t} \frac{\theta}{2N_f}.$$ (1)
where $t$ is the time coordinate and $N_f$ is the number of the light flavors in the theory.

The influence of the chiral imbalance on the thermal phase transition in the magnetic field background was addressed in Ref. [9] regarding possible applications to the chiral magnetic effect. Working in the Nambu–Jona-Lasinio (NJL) model coupled to the Polyakov loop (PNJL), the authors of Ref. [9] have found that the chiral imbalance makes the temperature of the chiral phase transition smaller while the strength of the transition becomes stronger. In our paper we confirm the findings of Ref. [9] working at zero magnetic field in the linear sigma model.

The quark part of the Lagrangian (3),

$$L_q = \bar{\psi} \gamma^\mu (\partial_\mu - iA_\mu) \psi + \gamma^5 \tau \cdot \bar{\psi} \gamma^0 \psi,$$  

provides the interaction between the quarks $\psi$, the chiral fields $\sigma$, $\bar{\sigma}$, and the field $A_\mu$ via the covariant derivative $\gamma^\mu (\partial_\mu - iA_\mu)$. The Lagrangian (4) also includes the real-valued chiral chemical potential $\mu_s$.

The dynamics of the chiral fields is described by the second term in the Lagrangian (3),

$$L_\sigma(\sigma, \bar{\sigma}) = \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \bar{\sigma} \partial^\mu \bar{\sigma}) + \partial_\mu \pi^+ \partial^\mu \pi^- - V_\sigma(\sigma, \bar{\sigma}),$$  

where we have introduced charged and neutral mesons,

$$\pi^\pm = \frac{1}{\sqrt{2}} (\pi^1 \mp i \pi^2), \quad \pi^0 = \pi^3,$$

respectively. The first and the second terms of the potential,

$$V_\sigma(\sigma, \bar{\sigma}) = \frac{\lambda}{4} (\sigma^2 + \bar{\sigma}^2 - \nu^2)^2 - h\sigma,$$

provide, respectively, (strong) spontaneous and (weak) explicit breaking of the chiral symmetry. The phenomenologically acceptable parameters in Eqs. (4) and (7) are: $g = 3.3$, $\lambda = 20$, and $\nu = 87.7$ MeV [18]. We work in a mean field approximation, thus neglecting quantum fluctuations (i.e., the kinetic terms) of the scalar fields $\sigma$ and $\bar{\sigma}$.

Effects of the color confinement are encoded in the last term of the Lagrangian (3), which describes the potential of the Polyakov loop:

$$L_\Phi = -V_\Phi(\Phi, \bar{\Phi}, T).$$

As in the previous case involving the mesonic chiral fields, we neglect possible kinetic terms for the Polyakov loop and consider only the potential term, following Ref. [19]:

$$\frac{V_\Phi(\Phi, \bar{\Phi}, T)}{T^4} = -\frac{1}{2} a(T)\Phi \bar{\Phi} + b(T) \times \ln(1 - 6\Phi \bar{\Phi} + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi \bar{\Phi})^2),$$

where

$$a(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2, \quad b(T) = b_3 \left( \frac{T_0}{T} \right)^3,$$

and $T_0 = 270$ MeV is the temperature of the deconfinement phase transition in the pure $SU(3)$ Yang-Mills theory without quarks [20]. Another choice of the critical temperature can be made by rescaling the parameter $T_0$ from 270 to 190 MeV in order to make a closer match with available lattice data [17]. The coefficients in Eq. (10) are:

$$a_0 = 16 \pi^2 / 45 = 3.51, \quad a_1 = -2.47, \quad a_2 = 15.2, \quad b_3 = -1.75.$$  

The Polyakov loop potential (9) respects the center $Z_3$ symmetry, $\Phi \to e^{i\pi n/3} \Phi$ with $n = 0, 1, 2$.

**III. THERMODYNAMICS AND PHASE DIAGRAM**

The thermodynamic potential per unit volume is:

$$\Omega = - \frac{T}{V_{3d}} \ln Z,$$  

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The first term in Eq. (14) corresponds to the divergent energy of the Dirac sea. After a proper regularization, the contribution does not change the qualitative physical picture. We ignore the logarithmic corrections as well, referring vacuum expectation value of the Polyakov loop to be fixed by the minimization of the thermodynamic potential (13) corresponds to the real values of the (traced) Polyakov loop \( \Phi \). We found that, in the presence of the chiral chemical potential \( \mu_s \neq 0 \) (and in the absence of the baryon chemical potential \( \mu_b \)), the minimum is always reached at \( \Phi = \Phi^0 \), contrary to the \( \mu \neq 0 \) case.

The fermion determinant breaks the center \( Z_3 \) symmetry of the Polyakov loop potential, and the global minimum of the thermodynamic potential (13) corresponds to the real-valued Polyakov loop. Moreover, one can show numerically that the presence of the charged pion condensates \( \pi^\pm \) makes the thermodynamic potential larger so that these condensates are disfavored, \( \langle \pi^\pm \rangle = 0 \). The last term in Eq. (7) forces the neutral pion condensate to be zero, \( \langle \pi^0 \rangle = 0 \).

Therefore we are left with two unknown expectation values which are to be fixed by the minimization of the thermodynamic potential. These are the real part of the Polyakov loop, \( \Re \Phi \), and the chiral order parameter \( \sigma \).

The expectation values of the Polyakov loop and of the chiral order parameter \( \sigma \) are shown as functions of the temperature in Figs. 1 and 2, respectively. The chiral field is normalized to unity at \( T = \mu_s = 0 \), and the corresponding vacuum expectation value \( \sigma_0 \) is given by the pion decay constant:

\[
\sigma_0 = \langle \sigma \rangle_{T=\mu_s=0} = f_\pi (T=\mu_s=0) = 92.2 \text{ MeV}.
\]  

The temperature and chiral chemical potential in Figs. 1 and 2 are expressed in units of the critical transition temperature \( T_c(0) \) at zero chiral chemical potential in \( \text{PLSM}_q \):

\[
T_c(0) \equiv T_c(\mu_s=0) = 213.2 \text{ MeV} \quad \text{[in \( \text{PLSM}_q \).]}
\]  

Because of the presence of the explicit symmetry-breaking term in the chiral potential \( V_{\sigma^i} \), Eq. (7), the transition between the quark-gluon plasma phase and the

\[
\int_0^\infty dp \int_{-1}^1 d\alpha \left[ n_{\alpha,s}^q (p) + n_{\alpha,s}^\bar q (p) \right] \frac{\partial \omega_s (p)}{\partial p},
\]  

where \( n_{\alpha,s}^q \) and \( n_{\alpha,s}^\bar q = n_{\alpha,s}^q \) are, respectively, the real-valued occupation numbers (summed over colors) for quarks and antiquarks carrying the helicity \( s \), and

\[
n_{\alpha,s} = \frac{3e^{-\omega_s/T} (\Phi + 2\Phi e^{-\omega_s/T} + e^{-2\omega_s/T})}{1 + 3(\Phi + \Phi e^{-\omega_s/T}) e^{-\omega_s/T} + e^{-3\omega_s/T}}.
\]  

The mean field values of the fields \( \sigma, \tilde{\sigma}, \) and \( \Phi \) are found by a (numerical) minimization of the thermodynamic potential given by Eqs. (13), (7), (9)–(11), and (16) with respect to the variations of \( \sigma, \tilde{\sigma}, \) and \( \Phi, \tilde{\Phi} \), Eq. (2), at fixed values of the temperature \( T \) and chiral chemical potential \( \mu_s \). The minimum of the thermodynamic potential is always reached at the real values of the (traced) Polyakov loop \( \Phi \). We found that, in the presence of the chiral chemical potential \( \mu_s \neq 0 \) (and in the absence of the baryon chemical potential \( \mu_b \)), the minimum is always reached at \( \Phi = \Phi^0 \), contrary to the \( \mu \neq 0 \) case.

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\[
n_{\alpha,s} = \frac{3e^{-\omega_s/T} (\Phi + 2\Phi e^{-\omega_s/T} + e^{-2\omega_s/T})}{1 + 3(\Phi + \Phi e^{-\omega_s/T}) e^{-\omega_s/T} + e^{-3\omega_s/T}}.
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of the chiral chemical potential and deconfinement phase transitions are tightened to each other. The chiral chemical potential increases. Moreover, the chiral gluon plasma phase and the hadron phase decreases as the critical temperature of the transition between the quark-gluon plasma phase and the hadron phase at zero chemical potential is a smooth crossover; the solid line corresponds to the first order phase transition; and the [red] point marks the second order critical end point (22). The inset shows the same phase diagram with the chiral chemical potential axis plotted in the logarithmic scale.

\[ n_5 = n_R - n_L = \frac{\langle N_q \rangle}{V_{3d}} = \langle \bar{\psi} \gamma^i \gamma^5 \psi \rangle = -\frac{\partial \Omega_q}{\partial \mu_5}, \tag{21} \]

should in general be nonzero at a nonvanishing chiral chemical potential (and, naturally, \( n_5 = 0 \) at \( \mu_5 = 0 \)).

We plot (the one-third power of) the density of the chiral charge (21) in Fig. 3. The chiral imbalance of the media in the presence of a fixed chemical potential \( \mu_5 \) is a growing function of the temperature. The chiral charge density \( n_5 \) is strongly enhanced at the transition temperature \( T_c = T_c(\mu_5) \), as the system goes from the hadron phase to the quark-gluon plasma phase.

The phase diagram in the \( (\mu_5, T) \) plane is shown in Fig. 4.

The transformation of the crossover transition (the dashed line at lower values of \( \mu_5 \)) to the first order phase transition (the solid line at larger \( \mu_5 \)) goes via a critical end point (CEP) in which the transition becomes a second order transition. We found that the CEP (the [red] point) lies on the quark-gluon plasma phase.

\[ (\mu_5, T)^{\text{CEP}} = (0.232 T_c^{(0)}, 0.998 T_c^{(0)}). \tag{22} \]

Our phase diagram, Fig. 4, computed in the PLSMq at zero magnetic field agrees qualitatively with the corresponding phase diagram of Ref. [9], where the calculations are done in the PNJL model in a weak magnetic field background\(^1\) in the window \( 0 \leq \mu_5 \leq 2.3 T_c^{(0)} \). The critical curves \( T_c = T_c(\mu_5) \) in the PLSMq and PNJL models are quite close to each other, while the positions of the corresponding CEPs differ from each other substantially, given that the location of the CEP in the PNJL model is independent of the strength of the magnetic field \( B \) in a weak field [9].

\(^1\)The critical temperature curve \( T_c = T_c(\mu_5, B) \) is almost independent of the strength of the magnetic field \( B \) in a weak field [9].
Notice, however, that parameters (for example, at large model of QCD, and therefore, at large values of massive chemical potential smoothly at asymptotically large values of the chiral potential [22].

It is important to mention that the line of the first order phase transition does not hit the $\mu_5$ axis at any finite value of the chiral chemical potential. Instead, the $\text{PLSM}_q$ model predicts that the $T_c = T_c(\mu_5)$ curve approaches this axis smoothly at asymptotically large values of the chiral chemical potential $\mu_5$:

$$\lim_{\mu_5 \to \infty} T_c(\mu_5) = 0. \quad (23)$$

Notice, however, that $\text{PLSM}_q$ is an effective low-energy model of QCD, and therefore, at large values of massive parameters (for example, at large $\mu_5$), the results coming from this model may become inaccurate. Therefore, the prediction in Eq. (23) should be considered with care.

**IV. CONCLUSIONS**

The chirally imbalanced hot quark-gluon plasma may emerge in heavy-ion collisions at RHIC and LHC experimental facilities. We have computed the QCD phase diagram in the chirally imbalanced background at finite temperature (Fig. 4) using the linear sigma model coupled to quarks and to the Polyakov loop (PLSM) at zero magnetic field. Our results are in qualitative agreement with conclusions of an earlier study of the QCD finite-temperature phase transition in the PNJL model in simultaneously imposed chiral and magnetic field backgrounds [9].

We have found that the increase of the chiral chemical potential $\mu_5$ tightens the link between deconfinement and chiral phase transitions, simultaneously lowering the common critical temperature and strengthening the order of the phase transition by converting the crossover (realized at low chiral imbalance) into the first order phase transition (found at higher chiral imbalance). The location of the second-order critical end point—at which the crossover turns into the first order phase transition—is given in Eq. (22). The $\text{PLSM}_q$ predicts that, at strictly zero temperature, the system always stays in the hadronic phase regardless of the value of the chiral chemical potential [Eq. (23) and Fig. 4].

We are mainly interested in the finite-temperature phase diagram of QCD at nonzero chiral chemical potential because the chiral magnetic effect is realized in the presence of the strong magnetic field in the chirally imbalanced background [4]. We confirm that some effects of the strong magnetic field and the chiral chemical potential on the critical temperature of the QCD phase transition are opposite: the magnetic field tends to increase the temperature of the phase transition [8–12] while the chirally imbalanced background forces the critical temperature to become lower [9]. Moreover, the magnetic field background splits the chiral and deconfinement phase transitions [10] (although the splitting can be small [11]), while the chiral imbalance tightens the link between these transitions. However, despite these dissimilarities, the external magnetic field and the chiral imbalance have one common feature: they both make the QCD phase transition stronger.

Finally, we would like to note that the predictions for the QCD phase diagram in the $(\mu_5, T)$ plane, given in Fig. 4 of the present article and in Ref. [9], can be directly tested from the first principles in numerical simulations of lattice QCD because the fermionic determinant with the chiral magnetic effect is realized in the presence of the strong magnetic field in the chirally imbalanced background [9].

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