On two generalisations of the final value theorem: scientific relevance, first applications, and physical foundations

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On two generalisations of the final value theorem: scientific relevance, first applications, and physical foundations

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The present work considers two published generalisations of the Laplace-transform final value theorem (FVT) and some recently appeared applications of one of these generalisations to the fields of physical stochastic processes and Internet queueing. Physical sense of the irrational time functions, involved in the other generalisation, is one of the points of concern. The work strongly extends the conceptual frame of the references and outlines some new research directions for applications of the generalised theorem.

Keywords: control; linear systems; queueing theory; Laplace transform; irrational functions; noninteger integration; ergodic systems

1. Introduction

The present work is motivated, on one hand, by the appearance of the applications (Walraevens, Fiems, Moeneclaey, and Bruneel 2007; Lapas 2008; Lapas, Morgado, Vainstein, Rubi, and Oliveira 2008; Walraevens, Fiems, and Moeneclaey 2009) of the generalisation (Gluskin 2003) of the Laplace transform final value theorem (FVT) (Zadeh and Desoer 1963; Doetsch 1974; Hayt and Kemmerly 1993), and, on the other hand, by the more recent work (Chen, Lundberg, Davison, and Bernstein 2007) devoted to another generalization of the classical FVT. In this article, we discuss the applications of both generalisations and explore the use of irrational functions in the last generalization. Since such functions are rare in system theory, it was found necessary to go deeper into their physical origination. This leads to some interesting analogies, which can encourage one to use such functions.

The work is composed of three parts. Section 2 describes some classical system situations to which the generalised FVT (GFVT) should be applied. Section 3 is devoted to the application of irrational functions to some control and signal processing problems. Finally, Section 4 is devoted to some proofs and the applications of the GFVT. The connections between the topics involved are summarised in Figure 1. Since some connections are best/naturally seen via physics, others via mathematics, our argumentation is both physical and mathematical.

The generalisations of Gluskin (2003) and Chen et al. (2007) are very different, and for clarity, the notation GFVT is usually used only for the generalization of Gluskin (2003), as shown in Figure 1. This in no sense decreases the importance of the suggestions of Chen et al. (2007), and in fact, the discussion of irrational functions influences the discussion of control systems and gives some support to the discussion of queueing theory in Section 4, where sequences with non-integer powers arise.

2. Application of GFVT to classical systems

2.1. The classical final value theorem

Consider the one-directional Laplace transform $F(s)$ of $f(t)$,

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt.$$  

The classical FVT states that

$$\lim_{s \to 0} sF(s) = f(\infty).$$  \hspace{1cm} (1)

When $f(t)$ arises as a solution function, depending on the structure of a physical or engineering system and its given input, Equation (1) becomes relevant to the dynamics of the system. Although function $f(t)$ can
arise in different systems, linear or nonlinear, the Laplace transform is especially effective for linear ones. Therefore, we consider only linear equations in this article, both ordinary and in partial derivatives.

2.2. Possible nonexistence of \( f(\infty) \)

The classical FVT requires the existence of \( f(\infty) \). Since in physical reality this is not always known beforehand, the case when \( f(\infty) \) does (potentially) not exist is most interesting to study. There are two main possibilities for the non-existence of \( f(\infty) \).

We first consider the suggestion of Chen et al. (2007) to generalise (1) to include the case when \( f(t) \to \infty \) as \( t \to \infty \) (and, respectively, \( s \to 0 \) in the \( s \)-domain). The tendency to infinity is understood as, for instance, the tendency of \( e^t \), or \( te^t \), but not of \( e^t \sin t \). Therefore, if we use the one-to-one map of \( f \) given by the equality \( f(z) = \varphi(t) \), then \( f(t) \to \infty \) is turned here into \( \varphi(t) \to 1 \), and \( f(t) \to -\infty \) into \( \varphi(t) \to -1 \). The ‘infinity’ can thus be transformed into a finite value.

Second, we consider the suggestion of the earlier work (Gluskin 2003) to use the time average over the whole interval \((0, \infty)\) instead of the final-value \( f(\infty) \). Defining the time average over finite interval as

\[
\langle f(t) \rangle_t = \frac{1}{T} \int_0^T f(t) dt,
\]

we have the time-average of interest as the limit for \( t \to \infty \):

\[
\langle f(t) \rangle \equiv \lim_{t \to \infty} \langle f(t) \rangle_t = \lim_{t \to \infty} \frac{1}{T} \int_0^T f(t) dt,
\]

We thus replace (1) by

\[
\lim_{s \to 0} sF(s) = \langle f(t) \rangle,
\]

which replaces the requirement of the existence of \( f(\infty) \) by the much weaker requirement of a boundedness of \( f(t) \), providing the existence of \( \langle f(t) \rangle \).

Clearly, Equation (2) is a generalisation of (1); if \( f(\infty) \) exists, \( \langle f(t) \rangle = f(\infty) \). Indeed, since in this case \( f(t) \) is almost all the time close to \( f(\infty) \), it is clear that

\[
\lim_{t \to \infty} \frac{1}{T} \int_0^T f(\mu) d\mu = f(\infty).
\]

The proof of (2), for the cases when \( f(\infty) \) does not exist, is given in Section 4.

2.3. Comments on (1) and (2)

(1) The sinusoidal \( f(t) \) is the simplest example (Hayt and Kemmerly 1993; Zadeh and Desoer 1963) warning against the careless use of (1), since for this function \( \lim_{s \to 0} sF(s) = 0 \), but \( f(\infty) \) does not exist. It is also the simplest supporting example for (2) since in the generalised case \( \lim_{s \to 0} sF(s) \) equals the zero time-average of the sinusoid.

(2) It needs to be stressed that the values of \( f(\infty) \) and of \( \langle f(t) \rangle \) are influenced only by the asymptotic behaviour of \( f(t) \) as \( t \to \infty \), and thus here \( f(t) \) may denote any function having the same asymptotic behaviour. This observation is important when approximations of the functions of interest are needed.

(3) We generally require that \( \lim_{s \to 0} sF(s) \) exists, and it is argued (Section 4.1) that at least for a function that possesses a simple spectral presentation, \( \lim_{s \to 0} sF(s) \) and \( \langle f(t) \rangle \) exist simultaneously.

(4) It is worth stressing that \( \langle f(t) \rangle \) does not exist for \( e^t \sin t \) and also for \( t^\alpha \sin t \) for \( \alpha \geq 1 \). The latter is seen from the equality,

\[
\langle f \rangle_t = \frac{1}{T} \int_0^T \mu^{\alpha} \sin \mu d\mu = -t^{\alpha-1} \cos t + \alpha \frac{1}{T} \int_0^T \mu^{\alpha-1} \cos \mu d\mu,
\]

obtained using integration by parts, which is correct for any \( \alpha > 0 \). The oscillations in \( \langle f \rangle \) do not vanish for \( \alpha \geq 1 \), as \( t \to \infty \).

(5) The practical use of average is associated, in particular, with the fact that any averaging means some low-pass filtering. This use of averaging is standard in signal processing (Hamming 1962), and is expected to be of interest for control problems where high-frequency noise, influencing the system input and its structure, has to be filtered. The topic is also
relevant to spatial filtering structures where the input and output variables are spatially distributed (see Gluskin 2004, 2005) and references therein.

2.4. The averaging of vectors

Consider the vector-function, denoted as \( \mathbf{x}(t) \) or as \( \bar{\mathbf{x}}(t) \),

\[
\bar{\mathbf{x}}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T,
\]

for which we can consider the limiting value \( \bar{\mathbf{x}}(\infty) = (x_1(\infty), x_2(\infty), \ldots, x_n(\infty))^T \).

We define the averaged vector as

\[
\bar{\mathbf{x}}(t) = \langle (x_1(t), x_2(t), \ldots, x_n(t))^T \rangle = \langle (x_1(0), x_2(0), \ldots, x_n(0))^T \rangle,
\]

assuming that all of the averaged components exist.

Then, for a constant matrix \([A]\) we have

\[
\langle [A]\bar{\mathbf{x}}(t) \rangle = [A]\bar{\mathbf{x}}(t),
\]

which, for \( \bar{\mathbf{x}}(\infty) \) existing, becomes, as \( t \to \infty \), the identity \([A]\bar{\mathbf{x}}(\infty) = [A]\bar{\mathbf{x}}(\infty)\).

Passing now on to time-dependent matrices relevant to linear time variant (LTV) systems, we can compare

\[
\lim_{t \to \infty} \langle [A(t)]\bar{\mathbf{x}}(t) \rangle
\]

to

\[
\langle [A(t)\bar{\mathbf{x}}(t)] \rangle.
\]

Considering each component of the vector \([A(t)]\bar{\mathbf{x}}(t)\) in view of (3), and then applying (4) to the whole vector, we see if the limit-vector (5) exists, then average (6) equals (5).

2.5. The use of the transfer matrix

Now consider the transfer matrix \([H(s)]\) of a general LTI system. We are interested in the time averages of the input \( \bar{u}(t) \) and output \( \bar{y}(t) \).

By the definition of \([H(s)]\),

\[
\bar{Y}(s) = [H(s)]\bar{U}(s),
\]

with \( \bar{Y}(s) \) and \( \bar{U}(s) \) Laplace transforms of \( \bar{y}(t) \) and \( \bar{u}(t) \). In view of GFVT (2), the time average of the \( \bar{u}(t) \) simply results in the time average of \( \bar{y}(t) \):

\[
\langle \bar{y}(t) \rangle = \lim_{s \to 0} s \bar{Y}(s) = \lim_{s \to 0} s[H(s)]\bar{U}(s) = [H(0)]\bar{u}(t).
\]

Since \([H(0)]\) is physically realized by ignoring all of the differential elements having memory, this matrix is real-valued.

Equality (7) includes the case \( \bar{y}(\infty) = [H(0)]\bar{u}(\infty) \).

For the existence of \( \bar{y}(\infty) \), it is sufficient to require the stability of the system and the existence of \( \bar{u}(\infty) \).

Figure 2 shows a scalar (one input and one output) circuit example. Here, \( v_{in}(t) \) is the input function and \( v_{out}(t) \) is
the output function. It is immediately seen that
\( H(0) = 1/3 \), that is, \( v_{\text{out}}(\infty) = (1/3)v_{\text{in}}(\infty) \) if \( v_{\text{in}}(\infty) \)
eq 0 exists, and, more generally, \( \{v_{\text{out}}(t) = (1/3)v_{\text{in}}(t)\} \), regardless of what are the values of the capacitances and inductances.

3. Irrational functions

We now turn to the suggestions of (Chen et al. 2007) to consider the cases of infinite \( f(\infty) \) and irrational functions. The first suggestion is supported by the fact that a solution of a differential equation, linear or not, often has an infinite final value. An example is the queueing process (Section 4), which is unstable, and tends to infinity. However, the suggestion to use irrational functions is more constructive for physical applications, and we focus on it in this section.

3.1. The non-integer degree

While analysing the cases when both \( f(\infty) \) and \( \lim_{n \to 0} sF(s) \) are infinite, work (Chen et al. 2007) employs formulae that are useful for obtaining asymptotic expressions for \( f(t) \) as \( t \to \infty \). In particular, it is used that

\[
\lim_{t \to \infty} \frac{f(t)}{t^\lambda} = \frac{1}{\Gamma(\lambda + 1)} \lim_{s \to 0} s^{\lambda+1} F(s), \quad \lambda > -1,
\]

where \( \Gamma(\cdot) \) is the gamma function and \( \lambda \) need not be an integer.

Denoting the common value of the right- and left-hand sides of (9) as \( K(\lambda) \), we have on the one hand

\[ f(t) \sim K(\lambda) t^\lambda, \quad t \to \infty, \]

and, on the other hand,

\[ F(s) \sim K(\lambda) \frac{\Gamma(\lambda + 1)}{s^{\lambda+1}}, \quad s \to 0. \]

Redefining \( f \) and \( F \) by dividing them both by \( K(\lambda) \), and keeping the same functional notations, we obtain the following result.

**Theorem 1:** Let \( F(s) \) be the Laplace transform of \( f(t) \). Then, in terms of the asymptotic features of these functions we have the equivalence \( (\approx) \)

\[ f(t) \sim t^\lambda, \quad t \to \infty \]

\[ \Leftrightarrow \quad \left( F(s) \sim \frac{\Gamma(\lambda + 1)}{s^{\lambda+1}}, \quad s \to 0 \right). \]

(10)

We can thus speak here only about the asymptotic parts of functions that need not be precisely equal to \( t^\lambda \) or \( \frac{\Gamma(\lambda + 1)}{s^{\lambda+1}} \) for all \( t \) and \( s \) respectively.

The following result is also needed.

**Theorem 2:** If for the ratio of the Laplace Transforms \( F(s) \) and \( G(s) \),

\[ \frac{F(s)}{G(s)} \sim s^{-\lambda} \]

with a non-integer \( \lambda \), then at least one of the respective originals, \( f(t) \) and \( g(t) \), is asymptotically irrational.

**Proof:** Assume that both \( f(t) \) and \( g(t) \) are rational. Then, as is known from the theory of Laplace Transforms, the functions \( F(s) \), \( G(s) \), and thus their ratio, are rational, which contradicts (11).

The question remains whether or not irrational functions, which cannot be introduced by the use of simple LTI circuits, are practically significant. We therefore now observe how such functions can appear in the analysis of physical systems.

3.2. The degrees characterising the irrationality

Assume that \( f(t) \) is some meaningful physical variable related to the description of a medium, and thus this variable is bounded. Then, the time average of \( df/dt \), which usually appears in the dynamic description of the medium, is zero. Indeed, since \( f(t) \) is bounded,

\[ \lim_{t \to \infty} t \int_0^t \frac{df}{d\mu} d\mu = \lim_{t \to \infty} t \int_0^t df = 0. \]

(12)

The boundedness of \( f(t) \) is even an excessive requirement for (12), since for, say \( f(t) \sim t^{1/2} \), (12) is still correct. For instance, for a random walk type of process, such that each step takes the same time, we have for the distance \( R_n \) from the origin after \( n \) steps that \( |R_n| \sim \sqrt{n} \) and thus, according to (12), \( |R_{n+1} - R_n| = 0 \), with the average taken over all
natural \( n \). Thus, (12) leads to physically interesting conclusions.

Next, assume that \( df/dt \) is also bounded, which is realistic for many physical processes. The GFVT can then be applied to \( df/dt \), which leads to

\[
\lim_{s \to 0} s^2 F(s) = 0. \quad (13)
\]

This formula allows us to set some assumptions regarding the asymptotic behaviour of \( F(s) \) as \( s \to 0 \) or, equivalently, regarding the asymptotic behaviour of \( f(t) \) as \( t \to \infty \), for such realistic processes. Let us consider, for simplicity, a power-law dependence:

\[
F(s) \sim s^{-\alpha}.
\]

Then (13) requires \( \alpha < 2 \). For instance, \( F(s) \sim s^{-3/2} \) is permitted, which means, according to (10), that \( f \sim t^{1/2} \) as \( t \to \infty \).

We note that for \( 1 < \alpha < 2 \) we obtain \( \lim_{s \to 0} s F(s) = \infty \), that is, an infinite \( \{f\} \), which is relevant to the discussion in Chen et al. (2007), and for \( \alpha > 1 \), we have \( \{f\} \) finite, which is relevant to the discussion in Gluskin (2003).

The hypothesis that we put here forward is that, in many physical problems, the only requirement \textit{a priori} is the boundedness of \( f \) and \( df/dt \). This hypothesis allows us to consider many problems with irrational \( f(t) \). However, since the rational case of \( F(s) \sim s^{-1} \) is included, this hypothesis is too general for a decision regarding existence of irrational functions in reality. We have to approach, the appearance of irrationality more constructively.

### 3.3. Non-integer integration and differentiation

Non-integer integration and differentiation (Manabe 1961; Courant 1964; Gelfand and Shilov 1964; Barnes and Allan 1966; Harty, Lorenzo, and Qammer 1995) is a source of irrational functions. Let us start by recalling some necessary basics. For \( n \in \mathbb{N} \), the formula

\[
\int_0^t d\mu_1 \int_0^{\mu_1} d\mu_2 \cdots \int_0^{\mu_{n-1}} f(\mu_n) d\mu_n = \frac{1}{(n-1)!} \int_0^t (t - \mu)^{n-1} f(\mu) d\mu \quad (14)
\]

illustrates the important fact that the solution of an LTI equation can be presented as a convolution. Indeed, the \( n \)-th-order derivative of each side equals \( f(t) \), or, alternatively explained, since \( 1/s^n \) is the Laplace transform of \( t^{n-1}/(n-1)! \), the transform of each side of (14) is \( (1/s^n)F(s) \). Thus, the \( n \)-th-order integration of \( f(t) \), i.e. the solution \( x(t) \) of the differential equation \( \frac{dx}{dt} = f(t) \) with zero initial conditions, is identical to the first-order integration with the kernel \( (t - \mu)^{n-1} \).

By using a non-integer \( \lambda \) (\( \Re \lambda > 0 \)) instead of the integer \( n \) in the convolution in the right-hand side of (14), and noting that the Laplace transform of \( t^{\lambda-1} \) is \( 1/s^\lambda \), the \( \lambda \)-th-order integration in (14) is generalised to a non-integer (fractional) \( \lambda \)-th-order integration,

\[
D^{-\lambda} f(t) = \frac{1}{\Gamma(\lambda)} \int_0^t (t - \mu)^{\lambda-1} f(\mu) d\mu, \quad (14a)
\]

and the non-integer differentiation \( D^\lambda f(t) \) is introduced as the inverse operation.

As an example, consider the non-integer differential equation

\[
\frac{d^{1/2} x}{d t^{1/2}} = t.
\]

According to (14a) we have

\[
x(t) = \frac{1}{\Gamma(1/2)} \int_0^t \frac{\mu}{\sqrt{t - \mu}} d\mu = \frac{1}{\sqrt{\pi}} \int_0^t \frac{1 - \mu}{\sqrt{\mu}} d\mu = \frac{4}{3\sqrt{\pi}} t^{3/2}.
\]

The obtained relation, \( x(t) \sim t^{3/2} \), might be expected from the consideration of the physical dimensions, since \( (d^{1/2} x)/(d t^{1/2}) = t \) can be rewritten for some small finite differences as \( (\Delta x)/(\Delta t)^{1/2} = t \), that is, \( \Delta x = t(\Delta t)^{1/2} \).

More generally, the integral in (14a) has the physical dimension of \( t^{1/2} \).

Thus, the integration and differentiation of non-integer order lead to irrational functions. However, the mathematical freedom in the choice of \( \lambda \) does not eliminate the engineering importance of the ascribing physical sense to these operations. Therefore, let us treat the example of Manabe (1961), showing that the degree \( s^{1/2} \) can appear in a physically meaningful problem, described by a linear partial differential equation (PDE), with a spatial degree of freedom involved.

### 3.4. The physical example

In this example, a semi-infinite cable having distributed (per unit length) resistance \( R \) and capacitance \( C \) is considered, that is, there are two independent variables, the distance \( x \) along the cable and the time \( t \). Contrary to Manabe (1961), we allow not only a voltage but also a \textit{current} source to be the driver at the beginning of the line, since in modern electronics current sources are commonly met.
The describing equations

\[ \frac{\partial v}{\partial x} = Ri \]  
\[ \frac{\partial i}{\partial x} = C \frac{\partial v}{\partial t} \]  

are the distribution versions of the usual equations \( v = Ri \) and \( i = Cd\theta/dt \) for lumped circuits, while the minuses before \( \frac{\partial}{\partial x} \) appear because the positively defined \( v(x, t) \) and \( i(x, t) \) are decreasing with the distance from the source that is placed at \( x = 0 \).

It is important to observe that (15a, b) yield a diffusion (or heat propagation) type of equation for \( i(x, t) \),

\[ \frac{\partial^2 i}{\partial x^2} = RC \frac{\partial i}{\partial t}, \]  

which means that the mathematical features of the process in the cable are typical for a diffusion process.

Diffusion-type processes are a wide class of natural processes and, in some problems, as the present one and those associated with radiation, we can speak about the diffusion of energy only. Such processes help us to observe the different problems where irrational functions can appear.

Since \( R \) and \( C \) are spatially distributed parameters, the expression \( RCx^2 \) (and not \( RC \) as for a lumped circuit) is measured in seconds. Thus \( RCx^2/t \) is a non-dimensional variable in terms of which the similarity of the cable problem to different diffusion systems or processes can be considered.

3.4.1. The unusual input impedance

Performing Laplace transforms with respect to \( t \) of (15a, b), we obtain for the transforms \( V(x, s) \) and \( I(x, s) \), the equations

\[ -\frac{dI}{dx} = CsV - \frac{dV}{dx} = RI. \]

Dividing the left equation by the right one, we obtain

\[ \frac{dI}{dV} = \frac{sCV}{R^T}, \]

from which (assuming that \( V \) and \( I \) can be zero only simultaneously) we get

\[ F^2 = \frac{sC}{R} V^2, \]

that is

\[ I \sim \sqrt{s}V. \]  

The current thus appears to be a derivative of order \( 1/2 \) of the voltage. According to Theorem 2, such a situation must yield an irrational dependence on time in at least one of the functions \( i(x, t) \) and \( v(x, t) \).

Relation (17) is noted in Manabe (1961), but (Manabe 1961; Harty, Lorenzo, and Qammer 1995) and some other works considering the semi-infinite cable do not mention the similarity of the electrical process in this system with diffusion-type phenomena. We emphasize that the reason for the irrationality is the additional (spatial) degree of freedom.

The solution of (16) is given by the convolution (Sidorov, Fedoryuk, and Shabunin 1985)

\[ i(x, t) = i(0, t) * \frac{(RC)^{1/2}x}{2\sqrt{\pi t^{3/2}}} e^{-\frac{Ri^2}{x}}, \]

where \( i(0, t) \) is the input current source, \( i(t) \). Observe that the unit of ampere is obtained because the integration by \( t \) in the convolution adds second to the dimension (while \( (RC)^{1/2} \) is non-dimensional). Having the form of a zero-state response, (18) corresponds to the case when the source is connected at \( t = 0 \) to the line without any current, i.e. \( i(0, 0) \equiv 0 \).

As the main point of this example, for any rational source-function \( i(0, t) \), the solution function \( i(x, t) \) is irrational. This is the feature of the cable as a diffusive system. For an input voltage source \( v(t) = v(0, t) \), the problem of the semi-infinite cable is solved similarly and the irrationality is also obtained. (See also (Manabe 1961).)

3.4.2. Comparison with other physical systems

That \( V/I \sim s^{-1/2} \) in (17), means that the input impedance of the semi-infinite cable in terms of the frequency response equals \( Z(\omega) = \omega^{-1/2} \). It is interesting to note, for comparison, that a lumped inductor \( L \) having an iron core composed of insufficiently thin layers, has \( Z_L(\omega) \sim \omega^{1/2} \). The latter is because the depth of penetration (or decay) of electromagnetic field into iron (the known skin effect) is proportional to \( \omega^{-1/2} \).

Since the thick layers are not completely filled by the magnetic field, the inductance is proportional to the depth of the penetration, and thus is frequency-dependent; which yields \( Z_L(\omega) = j\omega L \sim \omega^{1/2} \) (more precisely, \( |Z_L(\omega)| \sim \omega^{1/2} \), because for such a inductor, \( ReZ_L \) is also significant), instead of \( Z_L(\omega) \sim \omega \) for the usual inductor. Note that the decaying of electromagnetic field in any conductive medium (here of the inductor’s core) is also a diffusion-type process. Some works in which the irrationality of the type \( s^{1/2} \) is obtained in other physical systems are cited in Hartly et al. (1995).
The role of the spatial degree(s) of freedom, observed in the diffusion equation, is also well seen in the problem of Brownian movement; more generally, in the ‘1/f-noise’ systems problems. In such problems, the power spectrum of a time function, i.e. the square of $F(j\omega)$, or $F(s)$, is observed to be proportional to $1/s$, which means that $F(s)$ is proportional to $s^{-1/2}$. The time processes in such systems are associated with the spatial movement of some particles; the processes in these additional degrees of freedom cause the irrationality in the time-dependence.

A relevant physical outlook on the solution of such a spatial-temporally problem and the possible irrationality is obtained when we consider the role of time inversion, as follows.

### 3.4.3. The degree of irrationality and time-inversion

In the field of PDE, irrational functions are obtained not only for the diffusion-type equation, but also for the wave equation with a dissipation (friction) term. Consider

$$\frac{\partial^2 u}{\partial t^2} = a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial t},$$

where $a$ and $b$ are positive constants. Sidorov et al. (1985) treat this equation by describing the oscillations of a string with a damping given by $b$, showing that for any fixed $x$, the oscillations of the string, caused by a pinch, are decaying as $t^{-1/2}$.

The irrationality can thus be explained by the role of the terms in (19) and (16), which include $\frac{\partial}{\partial t}$, responsible for the dissipation of energy. Such a term is associated with the irreversibility of the dissipation process ($t \rightarrow -t$ changes the equation); reversibility would contradict the fact of the heating that causes the damping, that is, the law of increase in entropy.

Contrary to that, the undamped wave equation ($b = 0$) is not changed by $t \rightarrow -t$, and the undamped wave would propagate as it is with the time inversion in the back direction, which means the reversibility of the process.

It is most interesting to observe, however, how simply the mathematics ‘formalises’ the fact of irreversibility of the process described by such an equation. Since such factors as $t^{-1/2}$ or $t^{-3/2}$, appearing in the solution functions, include $\sqrt{t}$, the physical requirement of the realness of the solution function quite formally prohibits negative $t$, that is, time inversion. For this reason, we can also expect, for instance, such a degree as $t^{3/4}$, but not as $t^{1/3}$ or $t^{2/5}$, to arise in physics problems with irreversibility; e.g. in statistical theories, hydrodynamics and non-stationary problems of quantum mechanics.

The above arguments suggest, furthermore, to consider PDEs having a term which is not necessarily diffusive, but which reverses its sign with the inversion of time, for instance $t^3 \frac{\partial^2 u}{\partial t^2}$ or $t^3 \frac{\partial^2 \theta}{\partial x^2}$.

### 3.5. Ordinary equations with singular coefficients

For our last example, we return to ordinary (1D) differential equations, again dealing with functions of only time.

It is known from the analytical theory of ordinary (generally, complex) differential equations (Sidorov et al. 1985) that, if the coefficients in the equations are singular functions, the solution function can include non-integer degrees. For instance, seeking the solution of the equation,

$$\frac{d^2 f}{dt^2} + \frac{a}{t} \frac{df}{dt} + \frac{b}{t^2} f = 0,$$

with $a$ and $b$ constants of the form $t^r$, we obtain for $\lambda$ the characteristic equation $\lambda^2 + (a - 1) \lambda + b = 0$. The roots $\lambda_1$ and $\lambda_2$ (needed to form $f(t) = A t^{\lambda_1} + B t^{\lambda_2}$, with $A$ and $B$ constants) need not be integers. Thus, ordinary differential equations including only derivatives of integer order, but having singular coefficients, also can be a source of irrational functions.

We conclude that the realisation of irrational functions is an interesting research direction.

### 3.6. Some further comments on (Chen et al. 2007)

(1) Being concerned with composition of functions, one notes that the usual partial fraction expansion of $F(s)$, appearing in the theory of the LTI circuits (Zadeh and Desoer 1963; Hayt and Kemmerly 1993), relates to a composition of $f(t)$ using some linearly independent functions. The rational $\{F(s)\}$ form a (mathematical) field (van der Waerden 1955) since summation and multiplication operations, and the inverse operations of subtraction and division, always preserve the rational structure. Contrary to this, irrational functions do not form any linear space, because the terms responsible for irrationality can be mutually cancelled with additions or subtractions.

The irrational functions are treated in Chen et al. (2007) as some individual objects, and investigation of how such functions can compose each other, that is, how the sets of such functions can be used effectively for particular problems in control theory, should also be a natural research continuation of the line of Chen et al. (2007).

(2) Circuit specialists use the frequency response $H(j\omega)$ ($(H(s))$, $s \rightarrow j\omega$, $\omega \in \mathbb{R}$) only for systems possessing a finite and stable impulse response $h(t)$, when $h(\infty)$
exists and is finite. Thus, the cases when \( f(t) \) is the impulse response of a system have to be carefully separately considered as regarding application of the conclusions of Chen et al. (2007) related to \( f(t) \to \infty \).

(3) Results of Chen et al. (2007) relate only to continuous functions and should be generalised to the important discrete case, that is, to sequences and their z-transforms. Below, we encounter a discrete sequence \( f_n \to \infty \), with a fractional degree of \( n \).

4. Time-average instead of final value

In this section, some results relevant to the use of the GFVT are presented. We provide some proofs, discuss the discrete case and go into some applications.

4.1. Proofs

4.1.1. Periodic functions

Following (Gluskin 2003), let us first prove the GFVT for periodic functions that (when not constant) have no \( f(\infty) \), using the known formula for the Laplace transform of a \( T \)-periodic function, \( f(t + T) = f(t) \):

\[
sF(s) = \int_0^T e^{-st}f(t)dt = \frac{1}{1 - e^{-stT}} \int_0^T f(t)dt
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t)dt. \quad (20)
\]

At the last step of this derivation, we used that, for the periodic function \( \langle f(t) \rangle_T = \langle f(t) \rangle \).

It is obvious from (20) that, for any periodic function, \( \lim_{s \to 0} sF(s) \) and \( \langle f(t) \rangle \) exist simultaneously.

4.1.2. On the spectral composition of \( f(t) \)

We now can obtain (20) by employing a constant function together with sinusoidal functions, and using the Fourier expansion of any periodic function. This possibility is far-reaching, since then the use of the linearity of the operator \( sF(s) \) with regard to \( F(\cdot) \) and the approaching \( F(\cdot) \) as a composition allows us to consider any real-valued almost periodic function

\[
f(t) = \sum_k c_k e^{i\lambda_k t} \quad (21)
\]

with any \( \{\lambda_k\} \) (when complex, in conjugated pairs). For instance, the function \( A + B \sin \omega t + C \cos \sqrt{2} \omega t \) is included. Such functions can be relevant to different stochastic-processes applications, as those in Walraevens et al. (2007), Lapas (2008), Lapas et al. (2008) and Walraevens et al. (2009).

It is seen from the above construction that, when the spectral analysis of an \( f(t) \) (not necessarily periodic) is possible, then \( \lim_{s \to 0} sF(s) \) and \( \langle f(t) \rangle \) exist simultaneously, and for such functions the generalised theorem is also proved.

4.1.3. Comments

(1) While for the case of \( f(\infty) \) existing (and then also \( f^2(t) \to f(\infty)^2 \)), we have for the dispersion of \( f(t) \)

\[
\langle (f - \langle f \rangle)^2 \rangle = \langle f^2 \rangle - \langle f \rangle^2
\]

\[
= f(\infty)^2 - f(\infty)^2 = 0
\]

for \( f(\infty) \) not existing, any non-constant bounded asymptotic form of \( f(t) \), for instance that given by Equation (21), always provides a nonzero dispersion.

(2) For a bounded \( f(t) \), the non-existence of \( f(\infty) \) always means some oscillations of \( f(t) \).

(3) In many physical systems, large-scale time processes are not stabilised, and we can speak just about \( \langle f(t) \rangle \) or \( \langle f^2 \rangle \). An example can be, for instance, the intensity of wind velocity at a certain spatial point in the atmosphere. This parameter can be important, e.g. for wind stations generating electrical energy.

4.2. The discrete case

Using arguments similar to those that led to (2), Gluskin (2003) also suggests the discrete analogue of (2) for the z-transform \( F(z) \) of a sequence \( \{f_n, n \geq 0\} \),

\[
F(z) = \sum_{n=0}^{\infty} f_n z^{-n},
\]

namely

\[
\lim_{z \to 1+} (1 - z)F(z) = \langle f_n \rangle, \quad (2a)
\]

which generalises the classical discrete FVT (Lapas et al. 2008)

\[
\lim_{z \to 1+} (1 - z)F(z) = f_\infty. \quad (1a)
\]

Thus, sequences \( f_n \) not having a limit \( f_\infty \) can be considered. When \( f_\infty \) exists, \( \langle f_n \rangle = f_\infty \), i.e. (1a) is included in (2a), as is the case for continuous functions. However, many sequences have an average value \( \langle f_n \rangle \), but not a final value \( f_\infty \). For instance, the sequence \( \{1, -1, 1, -1, \ldots\} \) has a zero average but no limit value, and the same goes for \( \{1, 0, 0, 1, 0, 0, 1, \ldots\} \) with \( \langle f_n \rangle = 1/3 \). All these cases fit (2a).
4.3. Applications of the GFVT

Quoting Gluskin (2003), Walraevens et al. (2007) and Walraevens et al. (2009), apply (2a) to the theory of queueing (in particular, in the Internet), and Lapas (2008), and Lapas et al. (2008), apply (2) to the theory of Brownian motion. In all these works, long-term time processes are considered.

4.3.1. Brownian motion

The motivation in Lapas (2008) and Lapas et al. (2008) is associated with the ergodicity of some physical systems, that is, with the equality of statistical (ensemble) averages and time averages over the trajectory of a single particle. Allowing the calculation of a time average of a variable over a trajectory, the GFVT gives the statement of ergodicity a constructive meaning. Namely, a time function, for which the time average is treated as in Gluskin (2003), is associated in Lapas (2008) and Lapas et al. (2008) with such a trajectory, and then the ergodicity helps in calculating thermodynamic properties of a physical medium, which, by definition, are based on the ensemble averages, but are thus reduced to the time averages. Here we will not go into further details about this application of the GFVT to statistical physics.

4.3.2. Use of the GFVT in queueing theory

The application of the GFVT in Walraevens et al. (2007, 2009) relates to a queueing problem in the Internet. Since this application is also relevant to Chen (2007, 2009) relates to a queueing problem in the Internet. Since this application is also relevant to Chen (2007, 2009), it is therefore chosen to be considered in more detail.

Queueing theory is the mathematical study of queueing phenomena which occur in many daily applications. Modern applications of queueing theory are the design and operation of telecommunication and computer networks, such as the Internet.

In abstract terms, a queueing system is composed of one or more service units and a queue. The service units perform some kind of service to customers, whereas the queue is used by customers to wait until a service unit can service them. Customers arrive in the queueing system, wait in the queue until a service unit becomes available, and leave the queueing system (see Figure 3).

In the study of a particular queueing system, three steps are distinguished. First, the queueing system is transformed to a mathematical queueing model. This step encompasses the modelling of the arrival and service process. The arrival process describes the way in which customers arrive to the queueing system, while the service process summarises the way in which customers are serviced. It is important to note that we assume a stochastic framework. In general, all the variables are stochastic. Second, we determine the performance measures we want to calculate. Popular performance measures are the expected values of the number of customers in the system and of the sojourn time of a customer. Third, the queueing model is analysed.

The transform technique is one of the most popular analysis techniques. With this technique, the equations that relate the output to the input are transformed to the Laplace and/or z-transform domain, and solved in this domain. Finally, performance measures are calculated from the transforms.

To clarify the use of the (standard or generalised) FVT and GFVT in queueing theory, let us consider an example taken from Walraevens et al. (2009), introducing first some notations.

4.3.2.1. Some basic concepts for the modelling.

In the queueing model, time is discretised. It is divided into slots, all of equal length. The number of customer arrivals, \( e_k \), in the \( k \)-th slot is independent of the numbers of customer arrivals in other slots. The \( e_k \) \( (k \geq 1) \) have a common probability generating function (PGF) \( E(z) \), defined as \( E[z^e_k] \), with \( E[\cdot] \) the expected value operator of a stochastic variable. Note that \( E(z) \) characterises the arrival process completely. We further assume one service unit that services all the customers in a First-In-First-Out manner. When customers are present in the system at the beginning of a certain slot, the service unit starts to service the customer who is waiting the longest in the queue. This service ends at the end of the slot. As a final input, we characterise the sojourn time of the first arriving customer. A natural assumption is that the first customer arrives in a system with no customers present. He can then be served directly and his sojourn time is equal to exactly one slot. However, in Walraevens et al. (2009) we allow for a more general...
sojourn time of the first arriving customer. We merely assume that its PGF, \( D_1(z) \), is given.

The performance measure of interest is the expected sojourn time, \( E[d_n] \), of the \( n \)-th arriving customer in the system. More precisely, we want to know the number of slots the \( n \)-th arriving customer is expected to be present in the queueing system. The expected sojourn times can be regarded as a sequence \( \{f_n, n \geq 1\} \), with \( f_n = E[d_n] \). Since the calculation of the complete sequence \( \{f_n, n \geq 1\} \) is not straightforward, calculation of \( f_\infty \) (if it exists) or of \( \langle f_n \rangle \) is of great interest. This is where the (G)FVT comes into play.

4.3.2.2. Thus the mathematics works. In order to calculate the average expected sojourn time \( \langle f_n \rangle \), we start from the \( z \)-transform of the sequence \( \{f_n, n \geq 1\} \).

It is obtained in Walraevens et al. (2009) as

\[
F(z) = \frac{f_1 z}{1 - z} + \frac{E(z) - E(0) z}{1 - z^2} + \frac{(1 - E(0))(1 - z)(E(z) - 1)}{Y(z) - E(0) z D_1(Y(z))} + \frac{1}{(1 - E(0))(1 - z)Y(z)(1 - Y(z))},
\]

with \( f_1 \) the expected sojourn time of the first customer, \( E(z) \) the PGF of the number of arrivals in a slot, \( D_1(z) \) the PGF of the sojourn time of the first customer, and \( Y(z) \) the unique solution of the equation \( x - E(x) = 0 \), inside the complex unit disk of the \( x \)-plane for all \( z \), \( \mid z \mid < 1 \). Here, \( D_1(z), f_1 \) and \( E(z) \) are input functions and \( F(z) \) is the output function. Using the GFVT (Gluskin 2003) to the above expression for \( F(z) \), we can calculate \( \langle f_n \rangle \) explicitly as

\[
\langle f_n \rangle = \lim_{z \to 1} (1 - z)F(z) = \begin{cases} 1 + \frac{E'(1)}{E'(1) - E''(1)} & \text{if } E'(1) < 1, \\ \infty & \text{if } E'(1) \geq 1. \end{cases}
\]

(22)

Here, \( E'(1) \) and \( E''(1) \) are the first and second derivative of \( E(z) \) at \( z = 1 \). Note that L'Hopital's rule is used multiple times in the case \( E'(1) < 1 \), since \( E(1) = Y(1) = 1 \), by definition. Equation (22) shows that if \( E'(1) < 1 \) and \( f_\infty \) exists, the above formula yields the final value \( f_\infty \) of the sequence \( \{f_n, n \geq 1\} \).

In queueing-theoretic terms, \( f_\infty \) is called the steady-state value, or the expected sojourn time when the system has reached a steady state. This value can be found by the standard FVT. However, the GFVT is more generally applicable as it yields the average expected sojourn time if \( E'(1) < 1 \) also for \( f_\infty \) not existing (Gluskin 2003). Furthermore, it correctly shows the tendency to infinity if \( E'(1) \geq 1 \), which is relevant to Chen et al. (2007). In the latter case, we have an unstable queueing system. This can be understood as follows: on average more customers arrive in

the system than can be serviced. The number of customers waiting in the queue therefore increases unboundedly in time. The expected sojourn time of customers is also increasing and \( f_n \to \infty \) for \( n \to \infty \). Some illustrating examples for the three distinguished cases are shown in Figure 4.

5. Conclusions

Two generalisations of the FVT (Gluskin 2003; Chen et al. 2007) were considered. In both the cases, the FVT is extended to cases when the final value of the function of interest does not exist. Both extensions trigger some interesting research questions, which we have discussed in this article. For the line of Chen et al. (2007), we have suggested the consideration of compositions of irrational functions, the exploration of the physical origination of these functions and the extension to the discrete case. For the line of Gluskin (2003), we have focused on the role of transfer functions and on applications in the fields of classical and stochastic control. The whole field of possible applications of the GFVT is shown to be interesting and promising, especially for stochastic problems.

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Note

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