A GEOMETRIC PROOF OF THE UPPER BOUND ON THE SIZE OF PARTIAL SPREADS IN $H(4n + 1, q^2)$

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Abstract. We give a geometric proof of the upper bound of $q^{2n+1} + 1$ on the size of partial spreads in the polar space $H(4n + 1, q^2)$. This bound is tight and has already been proved in an algebraic way. Our alternative proof also yields a characterization of the partial spreads of maximum size in $H(4n + 1, q^2)$.

1. Introduction

A classical finite polar space is an incidence structure, consisting of the totally isotropic subspaces of a projective space with respect to a non-degenerate sesquilinear form or a non-degenerate quadratic form. All dimensions will be assumed to be projective from now on, and we will also refer to $m$-dimensional subspaces as simply $m$-spaces. In particular, the 0- and 1-dimensional subspaces of such a polar space are known as its points and lines, respectively. The generators are its subspaces of maximal dimension. A partial spread of a classical finite polar space is a set of generators with no two incident with a common point. If a partial spread actually partitions the point set of the polar space, it is said to be a spread.

The Hermitian variety $H(n, q^2)$ is a particular type of classical finite polar space, consisting of the subspaces in $\mathrm{PG}(n, q^2)$, the points of which all have homogeneous coordinates $(x_0, \ldots, x_n)$ satisfying the equation $x_0^{q+1} + \ldots + x_n^{q+1} = 0$. In this polar space, the generators are $(n-1)/2$-dimensional, if $n$ is odd, or $(n-2)/2$-dimensional, if $n$ is even, and the number of points is given by $|H(n, q^2)| = (q^{n+1} + (-1)^n)(q^n - (-1)^n)/(q^2 - 1)$. We refer to [4] for proofs and much more information on Hermitian varieties and polar spaces in general.

Thas [6] proved that in $H(2n + 1, q^2)$ spreads, or thus partial spreads of size $q^{2n+1} + 1$, cannot exist, which has made the question on the size of a partial spread in such a polar space, an intriguing question. Improved upper bounds on the size of partial spreads in $H(2n + 1, q^2)$ were proved in [2].

On the other hand, partial spreads of size $q^{n+1} + 1$ in $H(2n + 1, q^2)$ were constructed for all $n \geq 1$ in [1], by use of a symplectic polarity of the projective space $\mathrm{PG}(2n + 1, q^2)$, commuting with the associated Hermitian polarity. In the Baer subgeometry of points on which these two polarities coincide, a (regular) spread of the induced symplectic polar space $W(2n + 1, q)$ can always be found, and these

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q^{n+1} + 1 generators extend to pairwise disjoint generators of \( H(2n + 1, q^2) \). Maximality of partial spreads of \( H(2n + 1, q^2) \) constructed in this way was also shown for \( n = 1, 2 \) in [1] and for all even \( n \) in [5].

In [3], De Beule and Metsch proved that the maximum size of a partial spread in \( H(5, q^2) \) is \( q^3 + 1 \), and they also obtained additional information on partial spreads meeting that tight bound. In particular, they found that every generator of \( H(5, q^2) \), not meeting any element of such a partial spread \( S \) in a line or more, meets exactly \( q^2 - q + 1 \) elements of \( S \) in a point.

Using techniques from algebraic graph theory, we recently proved in [7] that the size of a partial spread in \( H(4n + 1, q^2) \) is at most \( q^{2n+1} + 1 \), and this bound is thus tight as well. It turns out that a geometric property of partial spreads of maximum size in \( H(5, q^2) \) can be generalized, and in fact paves the way for a new, completely geometric proof of the upper bound in \( H(4n + 1, q^2) \).

2. Tools

We first state a lemma by Thas [6].

**Lemma 2.1.** Let \( \pi_1, \pi_2 \) and \( \pi \) be three mutually disjoint generators in \( H(2n+1, q^2) \). The set of points on \( \pi_1 \), that are on a (necessarily unique) line of \( H(2n + 1, q^2) \) meeting both \( \pi \) and \( \pi_2 \), form a non-singular Hermitian variety in \( \pi_1 \).

**Corollary 2.2.** Let \( \pi_1, \pi_2 \) and \( \pi \) be three mutually disjoint generators in \( H(2n + 1, q^2) \). The number of generators meeting \( \pi \) in an \((n-1)\)-space, and meeting both \( \pi_1 \) and \( \pi_2 \) in a point is \( |H(n, q^2)| = \frac{(q^{n+1} + (-1)^n)(q^n - (-1)^n)}{q^2 - 1} \).

**Proof.** We let \( \perp \) denote the Hermitian polarity of PG\((2n + 1, q^2)\), associated with the polar space. It is obvious that every generator meeting \( \pi \) in an \((n-1)\)-space, can meet \( \pi_1 \) and \( \pi_2 \) in at most one point. On the other hand, through any point \( p_1 \in \pi_1 \), there is a unique generator \( \langle p_1, p_1^\perp \cap \pi \rangle \) meeting \( \pi \) in an \((n-1)\)-space. Hence we have to determine the number of points \( p_1 \in \pi_1 \) such that the generator \( \langle p_1, p_1^\perp \cap \pi \rangle \) also meets \( \pi_2 \) in a point.

First suppose that a point \( p_1 \in \pi_1 \) is such that the generator \( \langle p_1, p_1^\perp \cap \pi \rangle \) meets \( \pi_2 \) in a point \( p_2 \). In that case, the line \( p_1p_2 \) is a line of \( H(2n + 1, q^2) \), meeting \( \pi \) as well, as \( p_1^\perp \cap \pi \) is a hyperplane of \( \langle p_1, p_1^\perp \cap \pi \rangle \). Conversely, suppose a point \( p_1 \in \pi_1 \) is on a line of \( H(2n + 1, q^2) \), meeting \( \pi \) in \( p \) and \( \pi_2 \) in \( p_2 \). In that case, both \( p_1 \) and \( p \) are in the generator \( \langle p_1, p_1^\perp \cap \pi \rangle \), and hence so is the entire line \( p_1p \), including the point \( p_2 \). The desired result thus follows from Lemma 2.1. \( \square \)

3. The proof

**Theorem 3.1.** The size of a partial spread \( S \) in \( H(4n + 1, q^2) \), \( n \geq 1 \), is at most \( q^{2n+1} + 1 \). If \( |S| > 1 \) and \( \pi \in S \), then every generator meeting \( \pi \) in a \((2n-1)\)-space, will meet the same number of other elements of \( S \) in just a point, if and only if \( |S| = q^{2n+1} + 1 \). In that case, that number must be \( q^{2n} \).

**Proof.** Let \( S \) be a partial spread of size at least 2 in \( H(4n + 1, q^2) \). Consider a fixed element \( \pi \in S \). Let \( \{N_i | i \in I\} \) be the set of generators meeting \( \pi \) in a \((2n-1)\)-space. As the number of \((2n-1)\)-spaces in a generator equals \( (q^{4n+2} - 1)/(q^2 - 1) \), and the number of generators through any \((2n-1)\)-space in \( H(4n + 1, q^2) \) is given by \( q + 1 \), the cardinality of \( I \) is \( \frac{q^{4n+2} - 1}{q^2 - 1} - q \).
Note that any generator $N_i$ and any generator in $S \backslash \{ \pi \}$, are either disjoint or meet in a point. For every $N_i, i \in I$, let $t_i$ denote the number of generators in $S \backslash \{ \pi \}$, meeting $N_i$ in a point. We now count the number of pairs $(N_i, \pi')$, with $\pi'$ an element of $S \backslash \{ \pi \}$ meeting $N_i$ in a point, in two ways. As through every point $p'$ on an element $\pi'$ of $S \backslash \{ \pi \}$, there is a unique generator meeting $\pi$ in a point, we obtain:

\[(1) \quad \sum_{i \in I} t_i = (|S| - 1) \frac{q^{2n+2} - 1}{q^2 - 1}.\]

Now we count the number of ordered triples $(N_i, \pi_1, \pi_2)$, with $\pi_1$ and $\pi_2$ two distinct elements of $S \backslash \{ \pi \}$, both meeting $N_i$ in a point. We know from Corollary 2.2 that for every two distinct elements of $S \backslash \{ \pi \}$, there will be exactly $|H(2n, q^2)|$ generators $N_i$, meeting both of them in a point. Hence we obtain:

\[(2) \quad \sum_{i \in I} t_i (t_i - 1) = (|S| - 1)(|S| - 2) \frac{(q^{2n+1} + 1)(q^{2n} - 1)}{q^2 - 1}.\]

Combining (1) and (2), we find:

\[(3) \quad \sum_{i \in I} t_i^2 = (|S| - 1) \frac{q^{2n+1} + 1}{q^2 - 1} \left( (q^{2n+1} - 1) + (|S| - 2)(q^{2n} - 1) \right).\]

As $(\sum_{i \in I} t_i)^2 \leq (\sum_{i \in I} t_i^2)|I|$, with equality if and only if all $t_i$ are equal, this implies:

\[(|S| - 1)^2 \left( \frac{q^{4n+2} - 1}{q^2 - 1} \right)^2 \leq (|S| - 1) \frac{q^{2n+1} + 1}{q^2 - 1} \left( (q^{2n+1} - 1) + (|S| - 2)(q^{2n} - 1) \right) \frac{q^{4n+2} - 1}{q^2 - 1},\]

with equality if and only if all $t_i$ are equal. Since we assumed that $|S| > 1$, we can cancel factors on both sides to obtain:

\[(|S| - 1)q^{2n+1} - 1) \leq \left( (q^{2n+1} - 1) + (|S| - 2)(q^{2n} - 1) \right) q,\]

implying that $|S| \leq q^{2n+1} + 1$, with equality if and only if all $t_i$ are equal. In that case, their constant value must equal $(\sum_{i \in I} t_i)/|I| = q^{2n}$. \hfill \Box

4. Remark

This technique fails when applied to partial spreads in $H(4n + 3, q^2)$, where it yields a negative lower bound on the size instead.

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