Taylor-series expansions for queueing systems with train arrivals

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Abstract: We assess the performance of a discrete-time queueing system with train arrivals by means of Taylor-series expansion techniques.

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1. Introduction

Input traffic at various nodes in packet switched telecommunication networks typically exhibits various levels of correlation. As this significantly affects queueing performance, there is a continuous interest in analytically tractable queueing models with arrival correlation.

Train-arrival models, which are also referred to as session-based arrivals or queueing models with M/G/∞-input, are a particular type of two-level models: arriving packets are organized in larger entities denoted by sessions or trains. The model at hand entails a Markovian arrival model with a structured infinite state space, in the same class as the models considered a.o. in [1] and [2]. While train-arrival models come natural — the concept of sessions is omnipresent at higher layers of the network protocol stack — the tractability of an exact queueing analysis crucially depends on the assumption that the load associated to a single session equals or exceeds the service speed. Lifting this assumption poses some mathematical difficulties and is the subject of this paper.

To the best of our knowledge, the queueing problem at hand cannot be solved by exact analytical techniques. Therefore, heavy-traffic techniques have been applied to related arrival models [3]. In this paper, we focus on light-traffic approximations. In particular, we develop a Taylor-series expansion in the probability with which existing sessions terminate with a fixed probability and each active session produces a packet with a fixed probability . We refer to [5] for an overview of Taylor-series expansions in queueing systems.

2. Model description

We consider a discrete-time queueing system. Time is divided into fixed-length intervals, called slots. During slot , a number of new sessions starts denoted by the i.i.d. (independent and identically distributed) sequence of r.v.’s (random variables) with common probability generating function and let . Existing sessions terminate with a fixed probability , and each active session produces a packet with a fixed probability . We consider a single-server queue with a service rate of 1 packet per slot.

Let and denote the sequences of random variables representing respectively the number of active sessions and the buffer content during slot . They evolve according to the following system equation:

\[
\begin{align*}
X_{k+1} &= B_k^{(p)}(X_k) + N_{k+1} \\
U_{k+1} &= (U_{k-1}) + B_k^{(q)}(X_k)
\end{align*}
\]

where denotes a sequence of counting functions of independent and identically distributed Bernoulli processes, being the success probability. For a fixed , is a binomially distributed random variable with parameters and (the success probability).

Under the stability assumption , it can be shown by either Lounes-type of Lyapunov techniques, that there exists a unique stationary distribution for the process . We can show that the generating function satisfies the following functional equation:

\[
P(x, z) = \frac{1}{z} \left[ P(1-p+px(1-q+qz), z) + (z-1)P(1-p+px(1-q+qz), 0) \right] N(x(1-q+qz)).
\]

Despite its apparent simplicity, this functional equation cannot be solved in closed form for general and . Therefore, we focus on Taylor-series expansions.

3. Taylor series around \( q = 0 \)

By adapting existing results [4], we can ensure that the generating function is analytic in and in a neighbourhood of . Hence it is justified to expand thusly:

\[
P(x, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} M_{i,j}(x) z^j q^i,
\]

where our aim is to find the unknown functions for small . First we prove by induction that for and . Next, We plug the definition of Plugging the
definition of $M_{i,j}(x)$ into the functional equation, we find the following vector equation:

$$M_i(x) = M_i(1 - p + px)A_i(x) + N_i(x),$$

with $[M_i(x)]_j = M_{i,j}(x)$ and

$$N_i(x) = \sum_{k=0}^{i-1} \sum_{l=0}^{i} M_k^{(l)}(1 - p + px)B_{k,l}(x).$$

and $A_i$ and $B_{j,k}$ are known matrices, and $M_k^{(l)}$ is the $l$th derivative of $M_k$. In this way, we can obtain the desired $M_{i,j}(x)$ recursively.

Higher-order expressions are typically very long but can be easily calculated with a symbolic computer algebra system.

4. Other approximations

We can combine the Taylor expansion with other known results by using a Padé approximation. We discern two cases. If $\lambda < 1 - p$, then the system is stable for $q = 1$, for which the solution is known [2]. On the other hand, if $\lambda \geq 1 - p$, then the moments of the queue content diverge to $\infty$ for $q \rightarrow q_m = \frac{1 - \rho}{\lambda}$. In this case, we use heavy-traffic limit results, as pioneered by [6]. The performance measures are given in terms of the load $\rho$ and the asymptotic variance $V$, which is equal to

$$V = \lim_{\kappa \rightarrow \infty} \text{Var}[Y^{(\kappa)}_k] = \cdots = \frac{2p - 1}{\lambda} + \frac{\lambda_2}{\lambda^2}. \quad (4)$$

where $Y^{(\kappa)}_k$ denotes the number of arrivals in slots 1 to $\kappa$ that belong to a session that started in slot $k$.

In order to combine the Taylor expansion in the neighbourhood of $q = 0$ (light traffic) and the heavy-traffic results we use an idea proposed by [7]: as light-traffic approximations of the $n$th order provide the exact expressions for the first $k$ derivatives around $\rho = 0$, and heavy-traffic approximations provide the exact asymptote around $\rho = 1$, an expression of the form

$$E[U^{(p)}_{LT+HT}] = \theta_{1,n}q + \theta_{2,n}q^2 + \cdots + \theta_{k,n}q^k + \left(\frac{q\lambda}{1-p}\right)^{k+1}E[U^{(q)}_{HT}] \quad (5)$$

retains all the information contained in the two approximations.

5. Numerical evaluation

We provide a numerical example in which $\{S_k\}$ are Poisson distributed rv’s with mean $\lambda$.

Figure 1 depicts the mean queue content versus $q$. We assume $p = 0.99$ and $\lambda = 0.02$ such that the mean session duration is 100 slots and such that the load is 100% if sessions produce packets with probability $q_m = 0.5$. We assume $p = 0.9$ and $\lambda = 0.08$ such that the mean

Figure 1: LT+HT approximant for the mean queue content up to different orders as depicted versus $q$.

session duration is 10 slots and such that the load is 100% if sessions always produce packets ($q = 1$). Figure 1(a) depicts the various approximants as well as simulation results. To facilitate the display of the heavy-traffic results, we scale with $(1 - \rho)$.

References


