Comment on: a two-stage fourth-order "almost" $P$-stable method for oscillatory problems

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Abstract

$y''=f(x,y)$ is proposed. We claim that it is possible to retrieve this combination of multistep methods by means of
the theory of parameterized Runge–Kutta–Nyström (RKN) methods and moreover to generalize the method discussed by

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We first introduce a Runge–Kutta–Nyström method in parameterized form (1)–(3) [5]

$y_{n+1} = y_n + h y'_n + h^2 \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i)$, (1)

$y'_{n+1} = y'_n + h \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i)$ (2)

with

$Y_i = (1 - v_i) y_n + v_i y_{n+1} + (c_i - v_i - w_i) h y'_n + w_i h y'_{n+1} + h^2 \sum_{j=1}^{s} x_{ij} f(x_n + c_j h, Y_j)$ (3)

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An RKN method is therefore also characterized by the tableau

\[
\begin{array}{c|c|c|c}
\mathbf{c}^T & \mathbf{v}^T & \mathbf{w}^T & \mathbf{X}^T \\
\hline
\mathbf{b}^T & \mathbf{b}^T \\
\end{array}
\]  

whereby \( \mathbf{v} \) and \( \mathbf{w} \) are vectors with, respectively, the entries \( v_i \) and \( w_i \) \( (i = 1, 2, \ldots, s) \), and whereby \( \mathbf{X} \) is an \( (s \times s) \) matrix with elements \( x_{ij} \). The relation with the classical definition of an RKN method [2] is \( A = X + v \mathbf{b}^T + w \mathbf{b}^T \).

The method introduced in [1]

\[
y_{n+1} = y_n + h \left( \frac{1}{2} (y'_{n+1} + y'_n) - \frac{h^2}{12} (f(x_{n+1}, y_{n+1}) - f(x_n, y_n)) \right),
\]

\[
y'_{n+1} = y'_n + \frac{h}{6} \left( f(x_{n+1}, y_{n+1}) + 4 f \left( x_n + \frac{h}{2}, \tilde{y}_{n+1/2} \right) + f(x_n, y_n) \right),
\]

\[
\tilde{y}_{n+1/2} = \frac{1}{2} (y_{n+1} + y_n) - \frac{h}{8} (y'_{n+1} - y'_n)
\]

for computing oscillating solutions of special second-order initial-value problems, is of the type of extended single-step methods, where a combination is made of an extended trapezoidal formula, Simpson’s rule and a three-point formula for the mid-point. This can be rewritten in the form of the RKN method (1)-(3) in several ways, such as

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{48} & -\frac{1}{12} & -\frac{1}{48} & 1 & 0 \\
1 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & 0 & & \end{array}
\]  

or as

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{8} & 0 & 0 & 0 & & \end{array}
\]  

This method can be retrieved by imposing conditions on the order and stability of an \( s \)-stage RKN method in parameterized form [5].

In order to reconstruct the method of Chawla in form (8), we start by choosing

\[
c^T = (0, \frac{1}{2}, 1)^T.
\]
As the first and last stage correspond, respectively, to the previous and current integration point, the following can be motivated:

\[ v_1 = v_3 = 0, \quad x_{11} = x_{12} = x_{13} = 0 \quad \text{and} \quad x_{31} = \bar{b}_1, x_{32} = \bar{b}_2, x_{33} = \bar{b}_3. \]  

When all \( w \)-entries are chosen zero, the simplified expressions of the stages do no longer depend on the unknown \( y'_{n+1} \). If one requires the method to be of order four, which can be expressed by means of algebraical equations in terms of the coefficients of the method \[4\], all coefficients can be written in terms of two parameters \( x_{22} \) and \( x_{23} \).

The linear stability analysis of RKN methods is mainly determined by the notion of \( P \)-stability, which is explained in Definition 1. We start by introducing a matrix \( M(H) \) which arises when the RKN method is applied to the test equation \( y'' = -\lambda^2 y, \lambda \in \mathbb{R} \), and setting \( H = \lambda h \). Then the following equation is obtained:

\[
\begin{pmatrix}
  y_{n+1} \\
  h y'_{n+1}
\end{pmatrix}
= M(H)
\begin{pmatrix}
  y_n \\
  h y'_n
\end{pmatrix},
\]

where

\[
M(H) = \begin{pmatrix}
  1 - H^2 \bar{b}(I + H^2 A)^{-1} e & 1 - H^2 \bar{b}(I + H^2 A)^{-1} c \\
  -H^2 b(I + H^2 A)^{-1} e & 1 - H^2 b(I + H^2 A)^{-1} c
\end{pmatrix}.
\]

**Definition 1.** An RKN method is called \( P \)-stable when

\[
|M(H)| = 1, \quad 2 \pm \text{Trace}(M(H)) > 0
\]

is satisfied for all \( H > 0 \).

It should be remarked that also the original definition of \( P \)-stability as given by Lambert and Watson [3], leaves room for interpretation, in the sense that it is not precisely stated whether the equality sign in the trace condition in (14) should be withheld or not. When we restrict ourselves to Definition 1, a distinction has to be made with “almost” \( P \)-stable methods where a discrete number of \( H \)-values are excluded in the interval of periodicity.

“Almost” \( P \)-stability is only possible for \( x_{23} = -\frac{1}{48} \), where \( H^2 = 12 \) has to be excluded from the interval of periodicity. Hence the one-parameter family

\[
\begin{array}{cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{4} & 3x_{22} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & \frac{1}{48} + \frac{1}{2}x_{22} & x_{22} & \frac{1}{48} \\
\hline
& & & & \frac{1}{6} & \frac{1}{3} & 0 \\
& & & & \frac{1}{6} & \frac{1}{3} & \frac{1}{6}
\end{array}
\]

consists of fourth-order methods which are all “almost” \( P \)-stable. For the particular case of \( x_{22} = -\frac{1}{12} \), Chawla’s method (8) can be found.
Alternatively (15) can be written as

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{4} & -2w_3 & w_3 & \frac{1}{48} + \frac{1}{6}w_3 & -\frac{1}{48} - \frac{1}{6}w_3 & 0 \\
\frac{1}{6} & 0 & \frac{1}{3} &
\end{array}
\tag{16}
\]

where (9) is reached for \( w_3 = -\frac{1}{8} \).

**Definition 2.** A Runge–Kutta–Nyström method has stage-order \( r \) iff

\[
C(r): \quad A.c^q = \frac{c^{q+2}}{(q+2)(q+1)}, \quad q = 0, 1, 2, \ldots, r.
\tag{17}
\]

This means that the order of the internal stages is at least \( r + 2 \), i.e.

\[Y_i = y(x_n + c_ih) + O(h^{r+3}), \quad i = 1, 2, \ldots, s.\]

It is worth remarking that all methods in the generalized family (15) have stage-order (see Definition 2) three.

**References**


