The Cauchy-Kovalevskaya extension theorem in discrete Clifford analysis

Hilde De Ridder* Hennie De Schepper† Frank Sommen‡

June 7, 2011

Abstract

Discrete Clifford analysis is a higher dimensional discrete function theory based on skew Weyl relations. It is centered around the study of Clifford algebra valued null solutions, called discrete monogenic functions, of a discrete Dirac operator, i.e. a first order, Clifford vector valued difference operator. In this paper, we establish a Cauchy-Kovalevskaya extension theorem for discrete monogenic functions defined on the standard $\mathbb{Z}^m$ grid. Based on this extension principle, discrete Fueter polynomials, forming a basis of the space of discrete spherical monogenics, i.e. homogeneous discrete monogenic polynomials, are introduced. As an illustrative example we moreover explicitly construct the Cauchy-Kovalevskaya extension of the discrete delta function. These results are then generalized for a grid with variable mesh width h.

MSC Classification: 39A12, 30G35
Key Words: Cauchy-Kovalevskaya extension, discrete Clifford analysis

1 Introduction

The Cauchy-Kovalevskaya theorem (see e.g. [5, 12]) has a long history; for a detailed account we refer to [6]. In the traditional case of continuous variables the theorem, in its most simple setting, reads as follows.

Theorem 1.1 *If the functions $F, f_0, \ldots, f_{k-1}$ are analytic in a neighbourhood of the origin, then the initial value problem*

$$\begin{aligned} \partial_t^k h(\underline{x}, t) &= F(\underline{x}, t, \partial_t^i \partial_{\underline{x}}^\alpha h) \\ \partial_t^j h(\underline{x}, 0) &= f_j(\underline{x}), \quad j = 0, \ldots, k-1 \end{aligned}$$

has a unique solution which is analytic in a neighbourhood of the origin, provided that $|\alpha| + i \leq k$.

In the case where the differential operator involved is the Cauchy–Riemann operator, i.e. where the differential equation reduces to $\partial_t h = -i\partial_x h$ (with $k = 1, |\alpha| = 1, i = 0$), the theorem states that a holomorphic function in an appropriate region of the complex plane is completely determined by its restriction to the real axis. In the case of a harmonic function, where now $\partial_t^2 h = -\partial_x^2 h$ (with $k = 2, |\alpha| = 2, i = 0$), additionally the values of its normal derivative on the real axis should be given in order to determine it uniquely. In fact, the necessity of these restrictions as initial values becomes clear in the following construction formula for the holomorphic and harmonic Cauchy-Kovalevskaya (short: CK) extensions.

*Clifford Research Group, Faculty of Engineering, Ghent University, Belgium, hdr@cage.ugent.be
†Clifford Research Group, Faculty of Engineering, Ghent University, Belgium, hds@cage.ugent.be
‡Clifford Research Group, Faculty of Sciences, Ghent University, Belgium, fs@cage.ugent.be

Proposition 1.1 *If the function $f_0(x)$ is real-analytic in $|x| < a$, then*

$$F(z) = \exp{(iy\frac{d}{dx})}\,[f_0(x)] = \sum_{k=0}^{\infty} \frac{1}{k!} i^k y^k f_0^{(k)}(x)$$

is holomorphic in $|z| < a$ and $F(z)|_{\mathbb{R}} = f_0(x)$. If moreover $f_1(x)$ is real-analytic in $|x| < a$, then

$$G(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} y^{2j} \left(\frac{d}{dx}\right)^{2j} [f_0(x)] + \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} y^{2j+1} \left(\frac{d}{dx}\right)^{2j} [f_1(x)]$$

is harmonic in $|z| < a$ and $G(z)|_{\mathbb{R}} = f_0(x)$, while $\frac{\partial}{\partial y} G(z)|_{\mathbb{R}} = f_1(x)$.

A higher dimensional generalization of the theory of holomorphic functions in the complex plane, providing at the same time a refinement of harmonic analysis, is obtained in Clifford analysis, see e.g. [1, 4, 10, 11]. Clifford analysis focusses on the study of monogenic functions, i.e. Clifford algebra valued null solutions of the Dirac operator $\partial_{\underline{x}} = \sum_{\alpha=1}^{m} \mathbf{e}_\alpha \, \partial_{x_\alpha}$, where $(\mathbf{e}_1, \ldots, \mathbf{e}_m)$ is an orthonormal basis of $\mathbb{R}^m$ underlying the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. We refer to this setting as the Euclidean case, since the fundamental group leaving the Dirac operator $\partial_{\underline{x}}$ invariant is the orthogonal group $\mathrm{O}(m;\mathbb{R})$, which is doubly covered by the $\mathrm{Pin}(m)$ group of the Clifford algebra. The CK extension theorem in Euclidean Clifford analysis is a direct generalization to higher dimension of the complex plane case; it reads as follows.

Theorem 1.2 *If $f(x_2, x_3, \ldots, x_m)$ is real-analytic in an open set Ω' of $\mathbb{R}^{m-1}$ identified with $\{\underline{x} \in \mathbb{R}^m : x_1 = 0\}$, then there exists an open neighbourhood Ω of Ω' in $\mathbb{R}^m$ and a unique monogenic function F in Ω such that its restriction to Ω' precisely is f. If moreover Ω' contains the origin, then in an open neigbourhood of the origin this CK-extension F is given by*

$$F(x_1, x_2, \ldots, x_m) = \exp\left(x_1 \mathbf{e}_1 \partial_{\underline{x}}{}'\right)[f] = \sum_{k=0}^{\infty} \frac{1}{k!} x_1^k (\mathbf{e}_1 \partial_{\underline{x}}{}')^k [f]$$

where $\partial_{\underline{x}}{}'$ stands for the restriction of $\partial_{\underline{x}}$ to $\mathbb{R}^{m-1}$.

Recently, in [2, 3, 9], a framework for a discrete counterpart of Euclidean Clifford analysis was set up, and it was further developed based on the introduction of skew Weyl relations in [7]. Definitions were given for a discrete Dirac operator D, discrete monogenic functions and discrete spherical monogenics, i.e. homogeneous discrete monogenic polynomials, defined as the eigenfunctions of a discrete Euler operator. Moreover, some basic results of discrete function theory, such as a Cauchy representation formula for discrete monogenic functions, were obtained. The main aim of this paper is to establish a CK extension theorem for discrete monogenic functions and, in particular, to apply it for the construction of bases for the spaces of discrete spherical monogenics. To make the paper self-contained an introductory section on discrete Clifford analysis is included.

2 Preliminaries of discrete Clifford analysis

In the discrete Clifford setting we consider the natural graph corresponding to the equidistant grid $\mathbb{Z}^m$ with orthonormal basis $\mathbf{e}_j$, $j = 1, \ldots, m$. We then introduce the traditional one-sided forward and backward differences $\Delta_j^{\pm}$, $j = 1, \ldots, m$, respectively acting on a function u as

$$\begin{aligned} \Delta_j^+[u] &= \frac{u(\cdot + h\mathbf{e}_j) - u(\cdot)}{h} \\ \Delta_j^-[u] &= \frac{u(\cdot) - u(\cdot - h\mathbf{e}_j)}{h} \end{aligned}$$

where h denotes the mesh width, which in the standard case considered here, will be $h = 1$. So from now, let

$$\Delta_j^+ u = u(\cdot + \mathbf{e}_j) - u(\cdot), \quad \Delta_j^- u = u(\cdot) - u(\cdot - \mathbf{e}_j)$$

A Clifford vector $\underline{x}$ will, in the current setting, only show integer co-ordinates. With respect to the $\mathbb{Z}^m$-neighbourhood of $\underline{x}$, the usual definition of the discrete Laplacian then explicitly reads

$$\Delta^*[f](\underline{x}) = \sum_{j=1}^m \left[\Delta_j^+ [f] (\underline{x}) - \Delta_j^- [f] (\underline{x})\right] = \sum_{j=1}^m [f(\underline{x} + \mathbf{e}_j) + f(\underline{x} - \mathbf{e}_j)] - 2mf(\underline{x})$$

the notation Δ^* referring to this operator being called the "star Laplacian", since it contains function values at the midpoints of the faces of the unit cube centered at $\underline{x}$.

The discrete Dirac operator factorizing this star Laplacian is introduced using the so-called Hermitean setting. In this setting each basis element $\mathbf{e}_j$ is split into two basis elements $\mathbf{e}_j^+$ and $\mathbf{e}_j^-$, cf. [9, 2, 3], and we consider the free algebra over $\{\mathbf{e}_j^+, \mathbf{e}_j^-\}$, satisfying the following relations:

$$\begin{aligned}
\mathbf{e}_j^- \mathbf{e}_\ell^- + \mathbf{e}_\ell^- \mathbf{e}_j^- &= 0 \\
\mathbf{e}_j^+ \mathbf{e}_\ell^+ + \mathbf{e}_\ell^+ \mathbf{e}_j^+ &= 0 \\
\mathbf{e}_j^+ \mathbf{e}_\ell^- + \mathbf{e}_\ell^- \mathbf{e}_j^+ &= \delta_{j\ell}
\end{aligned}$$

Observe that these conditions imply that $\mathbf{e}_j^2 = +1$, $j = 1, \ldots, m$, in contrast to the usual Clifford setting where traditionally $\mathbf{e}_j^2 = -1$ is chosen. To introduce the Dirac operator, one then combines each difference, forward or backward, with the corresponding forward or backward basis vector, i.e.

$$D = \sum_{j=1}^m \mathbf{e}_j^+ \Delta_j^+ + \mathbf{e}_j^- \Delta_j^- \tag{1}$$

It can be readily checked that the resulting Dirac operator indeed factorizes the star Laplacian: $D^2 = \Delta$. Considering the differences $\Delta_j^\pm$ $(j = 1, \ldots, m)$ as lowering operators, we then introduce, see [7], the raising operators $X_j^\pm$ $(j = 1, \ldots, m)$ satisfying the following "skew" Weyl relations:

$$\Delta_j^+ X_j^+ - X_j^- \Delta_j^- = 1 \tag{2}$$

$$\Delta_j^- X_j^- - X_j^+ \Delta_j^+ = 1 \tag{3}$$

which replace the classical Weyl relations holding in the continuous case for the partial derivatives and the vector variable. The traditional vector variable corresponding to the Dirac operator is then replaced by the operator

$$X = \sum_{j=1}^m \mathbf{e}_j^+ X_j^- + \mathbf{e}_j^- X_j^+$$

of which the components $X_j^\pm$ are no longer independent, but they are interconnected by (2)–(3).

Next, the discrete Euler operator E, see also [7], is defined by imposing the intertwining relation

$$DX + XD = 2E + m$$

which holds for the Dirac operator and the vector variable in the continuous Clifford case. This translates into the following explicit form:

$$E = \sum_{j=1}^m \mathbf{e}_j^+ \mathbf{e}_j^- X_j^- \Delta_j^- + \mathbf{e}_j^- \mathbf{e}_j^+ X_j^+ \Delta_j^+$$

Moreover, this discrete Euler operator E is then easily seen to satisfy also the usual intertwining relations with the Dirac operator and the vector variable respectively, i.e.:

$$DE = ED + D, \qquad EX = XE + X \tag{4}$$

The notion of homogeneity of a discrete polynomial is then defined as follows.

Definition 2.1 *A discrete polynomial P is called homogeneous of degree k if and only if it is an eigenfunction with eigenvalue k of the discrete Euler operator: $EP = kP$.*

In combination with (4), this definition implies that application of the operator X to a discrete homogeneous polynomial of degree k will result in a discrete homogeneous polynomial of degree $k+1$.

Introduction of the co-ordinate variable $\xi_j = X_j^+ \mathbf{e}_j^- + X_j^- \mathbf{e}_j^+$, $j = 1, \ldots, m$, enables us to work simultaneously on the considered graph and on its dual. Likewise we also consider the co-ordinate difference operator $\partial_j = \mathbf{e}_j^+ \Delta_j^+ + \mathbf{e}_j^- \Delta_j^-$, $j = 1, \ldots, m$, decomposing in this way the discrete Dirac operator and the vector variable respectively as

$$D = \sum_{j=1}^{m} \partial_j, \qquad X = \sum_{j=1}^{m} \xi_j$$

On account of the skew Weyl relations (2)-(3) for the raising operators $X_j^\pm$ and the lowering operators $\Delta_j^\pm$, it is easily seen that ξ_j and ∂_j $(j = 1, \ldots, m)$ satisfy the Weyl relations

$$\partial_j \xi_j - \xi_j \partial_j = 1 \quad \text{and} \quad \partial_\ell \xi_j + \xi_j \partial_\ell = 0, \ \ell \neq j$$

Moreover, using the intertwining relation $EX = X(E+1)$, it directly follows that $E\xi_j = \xi_j(E+1)$, whence $\xi_j^k[1]$, i.e. natural powers of the operator ξ_j acting on the ground state 1, are the basic homogeneous discrete polynomials of degree k in the variable x_j, similar to the basic powers x_j^k in the continuous setting. In the following lemma, their fundamental properties are listed, see [7].

Lemma 2.1 *For all $k \in \mathbb{N}$ and $j, \ell = 1, \ldots, m$ we have*

$$\begin{aligned} \partial_j \, \xi_j^k[1] &= k \, \xi_j^{k-1}[1] \\ \partial_\ell \, \xi_j^k[1] &= 0, \ \ell \neq j \\ \partial_j \xi_j^{k_1} \xi_\ell^{k_2}[1] &= k_1 \, \xi_j^{k_1-1} \xi_\ell^{k_2}[1], \ \ell \neq j \end{aligned}$$

Moreover, for any two multi-indices $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$ with $|\alpha| = |\beta|$, it then holds that

$$\partial_1^{\alpha_1} \ldots \partial_m^{\alpha_m} \left(\xi_1^{\beta_1} \ldots \xi_m^{\beta_m} \right) [1] = \begin{cases} \alpha! & \text{if} \quad \alpha = \beta \\ 0 & \text{if} \quad \alpha \neq \beta \end{cases}$$

where we have put $\alpha! = \alpha_1! \, \alpha_2! \ldots \alpha_m!$.

Furthermore, a closed form for these polynomials is obtained in the following theorem.

Theorem 2.1 *The homogeneous discrete polynomials $\xi_j^k[1]$ are given by $\xi_j[1](x_j) = x_j \, (\mathbf{e}_j^+ + \mathbf{e}_j^-)$ and*

$$\xi_j^{2n+1}[1](x_j) = x_j \, (\mathbf{e}_j^+ + \mathbf{e}_j^-) \prod_{i=1}^{n} (x_j^2 - i^2) \tag{5}$$

$$\xi_j^{2n}[1](x_j) = \left(x_j^2 + n x_j (\mathbf{e}_j^+ \mathbf{e}_j^- - \mathbf{e}_j^- \mathbf{e}_j^+) \right) \prod_{i=1}^{n-1} (x_j^2 - i^2) \tag{6}$$

for $n = 1, 2, \ldots$ and $j = 1, \ldots, m$.

Proof
From the definition of ξ_j itself it follows that $\xi_j[1](x_j) = x_j\,\mathbf{e}_j$, $j = 1,\ldots,m$. Next, take $k > 1$, then it follows by induction that $\xi_j^k[1](0) = 0 = \xi_j^k[1](1)$, for $j = 1,\ldots,m$. The homogeneous polynomial $\xi_j^k[1](x_j)$ thus is to be found as the unique solution of the system

$$\left\{\begin{array}{lll} \partial_j \xi_j^k[1] & = & k\,\xi_j^{k-1}[1] \\ \xi_j^k[1](0) & = & 0 \\ \xi_j^k[1](1) & = & 0 \end{array}\right.$$

Explicit low order calculations then reveal the form (5) when $n = 2k+1$ and the form (6) when $n = 2k$, respectively, which may indeed be checked by direct calculation to submit to the above conditions for any $k \geq 1$ and any $j = 1,\ldots,m$. ■

A function defined on $\mathbb{Z}^m$ is then called (left) discrete monogenic in a domain $\Omega \subset \mathbb{Z}^m$ iff it satisfies in Ω the equation $Df = 0$, or, in other words, if it is a null solution for the left action of the discrete Dirac operator. Homogeneous polynomial null solutions of D are called spherical discrete monogenics. For further use, we still mention that the dimension of the space $\mathcal{M}_k^{(m)}$ of spherical discrete monogenics of degree k on $\mathbb{Z}^m$ was established in [7] to be

$$\dim(\mathcal{M}_k^{(m)}) = \frac{(k+m-2)!}{k!(m-2)!} \tag{7}$$

3 The discrete Cauchy-Kovalevskaya extension

In this section we deal with the following problem.

Let f be a discrete function in the variables $x_2,\ldots,x_m$, defined on the grid $\mathbb{Z}^{m-1}$ and taking values in the algebra over $\{\mathbf{e}_2^+,\mathbf{e}_2^-,\ldots,\mathbf{e}_m^+,\mathbf{e}_m^-\}$. Does there exist a discrete monogenic function F in the variables $x_1,\ldots,x_m$, defined on the grid $\mathbb{Z}^m$ and taking values in the algebra over $\{\mathbf{e}_1^+,\mathbf{e}_1^-,\ldots,\mathbf{e}_m^+,\mathbf{e}_m^-\}$, such that $F|_{x_1=0} = f$?

This problem will be called the discrete Cauchy-Kovalevskaya extension (or CK extension) problem. To obtain a positive answer to it, preferably by explicit construction, is important, since it will enable us to generate discrete monogenic functions starting from ordinary discrete ones.

Now, putting

$$F(x_1,x_2,\ldots,x_m) = \sum_{k=0}^{\infty} \frac{\xi_1^k[1](x_1)}{k!}\, f_k(x_2,\ldots,x_m)$$

with $f_0 = f$, it is directly seen, on account of (5)–(6), that the function F takes the correct values and satisfies $F|_{x_1=0} = f$. For F to be moreover discrete monogenic it must vanish under the action of the Dirac operator D, (1), which we rewrite as

$$D = \partial_1 + \sum_{j=2}^{m} \partial_j = \partial_1 + D'$$

In order to determine the coefficient functions f_k, $k = 1,2,\ldots$ in such a way that the condition $DF = 0$ is fulfilled, we proceed by direct calculation, invoking Lemma 2.1 for the action of ∂_j on

$\xi_1^k[1]$. Since ∂_1 only acts on $\xi_1^k[1]$ and D' anticommutes with $\xi_1[1]$ we obtain

$$0 = DF = (\partial_1 + D') \left(\sum_{k=0}^{\infty} \frac{\xi_1^k[1]}{k!} f_k \right) = \sum_{k=0}^{\infty} \frac{\xi_1^k[1]}{k!} f_{k+1} + \sum_{k=0}^{\infty} (-1)^k \frac{\xi_1^k[1]}{k!} D' f_k$$

resulting into the recurrence relation

$$f_{k+1} = (-1)^{k+1} D' f_k \tag{8}$$

All of the above can now be summarized into the following definition.

Definition 3.1 *The CK extension $CK[f]$ of a discrete function $f(x_2, \ldots, x_m)$ is the discrete monogenic function*

$$CK[f](x_1, x_2, \ldots, x_m) = \sum_{k=0}^{\infty} \frac{\xi_1^k[1](x_1)}{k!} f_k(x_2, \ldots, x_m) \tag{9}$$

where $f_0 = f$ and $f_{k+1} = (-1)^{k+1} D' f_k$.

Observe that the foregoing definition does not contain any conditions on the original function f. Indeed, from (5)-(6) it follows that

$$\begin{aligned} \xi_1^{2n+1}[1](x_1) &= 0 \qquad \text{for } n \geqslant |x_1| & (10) \\ \xi_1^{2n}[1](x_1) &= 0 \qquad \text{for } n \geqslant |x_1| + 1 & (11) \end{aligned}$$

implying that for every point $(x_1, \ldots, x_m)$ of the grid $\mathbb{Z}^m$, there exists an $N \in \mathbb{N}$ such that all but the first N terms of the series in (9) vanish, whence the series reduces to a finite sum in every point of $\mathbb{Z}^m$. Thus, for any discrete function $f(x_2, \ldots, x_m)$, its CK extension is well-defined on $\mathbb{Z}^m$. Moreover, it is unique, as is stated in the following theorem.

Theorem 3.1 *Let F be a discrete monogenic function defined on $\mathbb{Z}^m$, with $F|_{x_1=0} \equiv 0$. Then F is the null function.*

Proof
The discrete monogenicity of F explicitly reads as

$$\mathbf{e}_1^+ \Delta_1^+ F(\underline{x}) + \mathbf{e}_1^- \Delta_1^- F(\underline{x}) + \sum_{j=2}^{m} \left(\mathbf{e}_j^+ \Delta_j^+ F(\underline{x}) + \mathbf{e}_j^- \Delta_j^- F(\underline{x}) \right) = 0$$

Now take $\underline{x} \in \mathbb{Z}^m$ with $x_1 = 0$. Since $F|_{x_1=0} \equiv 0$ the above expression reduces to

$$\mathbf{e}_1^+ F(\underline{x} + \mathbf{e}_1) - \mathbf{e}_1^- F(\underline{x} - \mathbf{e}_1) = 0 \tag{12}$$

Furthermore also $\Delta^* F = D^2 F = 0$, i.e.

$$\sum_{j=1}^{m} (F(\underline{x} + \mathbf{e}_j) + F(\underline{x} - \mathbf{e}_j)) - 2mF(\underline{x}) = 0$$

from which we obtain, again for $\underline{x} \in \mathbb{Z}^m$ with $x_1 = 0$,

$$F(\underline{x} + \mathbf{e}_1) + F(\underline{x} - \mathbf{e}_1) = 0 \tag{13}$$

Combination of (12) and (13) results into

$$(\mathbf{e}_1^+ + \mathbf{e}_1^-)F(\underline{x} - \mathbf{e}_1) = 0$$

whence $F(\underline{x} - \mathbf{e}_1) = 0$ and also $F(\underline{x} + \mathbf{e}_1) = 0$ for any $\underline{x}$ with $x_1 = 0$ and $(x_2, \ldots, x_m)$ arbitrary, implying that $F \equiv 0$ on $x_1 = 1$ and $x_1 = -1$. Repeating this procedure, we find that $F \equiv 0$ on $\mathbb{Z}^m$. ■

Corollary 3.1 (Uniqueness of the CK extension) *Let F_1 and F_2 be two discrete monogenic functions such that $F_1|_{x_1=0} \equiv f$ and $F_2|_{x_1=0} \equiv f$. Then F_1 and F_2 coincide.*

4 Discrete Fueter polynomials

The discrete CK extension procedure, as explained in the previous section, establishes a homomorphism between the space $\Pi_k^{(m-1)}$ of discrete homogeneous polynomials of degree k in $m-1$ variables and the space $\mathcal{M}_k^{(m)}$ of spherical discrete monogenics of degree k in m variables. This homomorphism is injective, as stated in Theorem 3.1 and Corollary 3.1. Moreover, a basis for the space $\Pi_k^{(m-1)}$ being given by the discrete homogeneous polynomials $\xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}$, $\alpha_2 + \ldots + \alpha_m = k$, its dimension is

$$\dim \Pi_k^{(m-1)} = \frac{(k + (m-1) - 1)!}{k!(m-2)!}$$

which exactly equals the dimension of $\mathcal{M}_k^{(m)}$, see (7), whence the homomorphism also is surjective. The CK extension procedure thus establishes an isomorphism between $\Pi_k^{(m-1)}$ and $\mathcal{M}_k^{(m)}$, allowing us to determine a basis for the space $\mathcal{M}_k^{(m)}$.

Definition 4.1 *Let $\underline{\alpha} \in \mathbb{N}^{m-1}$ with $\alpha_2 + \ldots + \alpha_m = k$. Then the discrete spherical monogenics*

$$V_{\underline{\alpha}} = CK\,[\xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}]$$

are called the discrete Fueter polynomials of degree k.

Theorem 4.1 *The set $\left\{V_{\underline{\alpha}} \,|\, \alpha_2 + \ldots + \alpha_m = k\right\}$ constitutes a basis for $\mathcal{M}_k^{(m)}$.*

Proof
The CK extension procedure constituting an isomorphism between both spaces, the basis

$$\{\xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m} \,|\, \alpha_2 + \ldots + \alpha_m = k\}$$

of $\Pi_k^{(m-1)}$ is transformed into the basis

$$\{CK\,[\xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}] \,|\, \alpha_2 + \ldots + \alpha_m = k\}$$

of $\mathcal{M}_k^{(m)}$. ■

Example 4.1 *The space $\mathcal{M}_2^{(3)}$ has dimension 3, on account of (7). A basis for it is given by the elements*

$$\begin{aligned} V_{(2,0)} &= CK[\xi_2^2] &= \xi_2^2 - 2\xi_1\xi_2 - \xi_1^2 \\ V_{(1,1)} &= CK[\xi_2\xi_3] &= \xi_2\xi_3 - \xi_1\xi_3 + \xi_1\xi_2 + \xi_1^2 \\ V_{(0,2)} &= CK[\xi_3^2] &= \xi_3^2 - 2\xi_1\xi_3 - \xi_1^2 \end{aligned}$$

of which it can be checked also directly that they are discrete monogenic, of homogeneity degree 2 in (x_1, x_2, x_3) and linearly independent.

The explicit construction of the discrete Fueter basis, in arbitrary dimension m and for arbitrary homogeneity degree k, is the subject of the forthcoming paper [8].

5 CK extension of the discrete delta-function

As an interesting and illustrative example we present the CK extension of the restriction to the $\mathbb{Z}^{m-1}$ grid of the function defined on $\mathbb{R}^{m-1}$ by

$$\delta_0(x_2,\ldots,x_m) = \begin{cases} 1 & (x_2,\ldots,x_m) = (0,\ldots,0) \\ 0 & (x_2,\ldots,x_m) \neq (0,\ldots,0) \end{cases}$$

This restriction, still denoted by δ_0, is usually called the discrete delta-function. Since every discrete function given by its values in the vertices of the grid can be written as a linear combination of shifted delta-functions, the CK extension of the delta-function will be a basic building block for the CK extension of other functions. Here we will treat only the case $m = 2$ explicitely; however, other dimensions may be directly computed as well.

Consider, for $m = 2$, the delta-function δ_0 in the variable x_2, i.e.

$$\delta_0(x_2) = \begin{cases} 1 & x_2 = 0 \\ 0 & x_2 \neq 0 \end{cases}$$

and, in general, for $k \in \mathbb{Z}$ arbitrary

$$\delta_k(x_2) = \delta_0(x_2 - k) = \begin{cases} 1 & x_2 = k \\ 0 & x_2 \neq k \end{cases}$$

Proposition 5.1 *The CK-extension of the delta-function $\delta_0(x_2)$ is well-defined in all points of the grid $\mathbb{Z}^2$, and given by*

$$CK[\delta_0](x_1,x_2) = \sum_{k=0}^{\infty} \frac{\xi_1^k[1](x_1)}{k!}\, f_k(x_2) \tag{14}$$

where $f_0 = \delta_0$ and

$$f_{2n} = \sum_{j=0}^{2n} (-1)^{j+n} \binom{2n}{j} \delta_{n-j} \tag{15}$$

$$f_{2n+1} = \mathbf{e}_2^+ \left(\sum_{j=0}^{2n+1} (-1)^{j+n+1} \binom{2n+1}{j} \delta_{j-n-1} \right) + \mathbf{e}_2^- \left(\sum_{j=0}^{2n+1} (-1)^{j+n} \binom{2n+1}{j} \delta_{n+1-j} \right) \tag{16}$$

Proof
>From the previous section it follows that the CK extension of δ_0 indeed takes the form (14), with $f_0(x_2) = \delta_0(x_2)$ and, according to (8), $f_{k+1}(x_2) = (-1)^{k+1} D' f_k(x_2)$. Here $D' = \mathbf{e}_2^+ \Delta_2^+ + \mathbf{e}_2^- \Delta_2^-$ and the action of $\Delta_2^\pm$ on $\delta_k(x_2)$ is given by

$$\begin{aligned} \Delta_2^+[\delta_k](x_2) &= \delta_k(x_2+1) - \delta_k(x_2) = \delta_{k-1}(x_2) - \delta_k(x_2) \\ \Delta_2^-[\delta_k](x_2) &= \delta_k(x_2) - \delta_k(x_2-1) = \delta_k(x_2) - \delta_{k+1}(x_2) \end{aligned}$$

The explicit expressions (15)–(16) then follow by an induction argument. ■

Now we determine the value of the CK extension of δ_0 in a point $(x_1, x_2) = (\ell_1, \ell_2)$ of the grid $\mathbb{Z}^2$. Since CK$[\delta_0]$ consists of products of delta-functions $\delta_k(x_2)$ with powers of ξ_1 (acting on the ground state), the value of the CK$[\delta_0]$ in (ℓ_1, ℓ_2) is given by the x_1-depending coefficient of δ_{ℓ_2} in the CK extension, evaluated in ℓ_1. We obtain the following result.

Theorem 5.1 *The value of the CK extension of the discrete delta function δ_0 in a point $(\ell_1, 0)$ of the grid $\mathbb{Z}^2$ is*

$$1 + \sum_{n=1}^{|\ell_1|} \frac{\xi_1^{2n}[1](\ell_1)}{(n!)^2} + \sum_{n=0}^{|\ell_1|-1} \frac{\xi_1^{2n+1}[1](\ell_1)}{n!\,(n+1)!} \left(\mathbf{e}_2^+ - \mathbf{e}_2^-\right) \tag{17}$$

Furthermore, its value in a point (ℓ_1, ℓ_2) with $\ell_2 > 0$ is

$$\begin{aligned}(-1)^{\ell_2-1} \frac{\xi_1^{2\ell_2-1}[1](\ell_1)}{(2\ell_2-1)!}\,\mathbf{e}_2^- &+ (-1)^{\ell_2} \sum_{n=\ell_2}^{|\ell_1|} \frac{\xi_1^{2n}[1](\ell_1)}{(n-\ell_2)!\,(n+\ell_2)!} \\ &+ (-1)^{\ell_2} \sum_{n=\ell_2}^{|\ell_1|-1} \frac{\xi_1^{2n+1}[1](\ell_1)}{(n-\ell_2)!\,(n+\ell_2)!} \left(\frac{\mathbf{e}_2^+}{n+\ell_2+1} - \frac{\mathbf{e}_2^-}{n-\ell_2+1}\right)\end{aligned} \tag{18}$$

while its value in a point (ℓ_1, ℓ_2) with $\ell_2 < 0$ is

$$\begin{aligned}(-1)^{\ell_2} \frac{\xi_1^{2|\ell_2|-1}[1](\ell_1)}{(2\,|\ell_2|-1)!}\,\mathbf{e}_2^+ &+ (-1)^{\ell_2} \sum_{n=|\ell_2|}^{|\ell_1|} \frac{\xi_1^{2n}[1](\ell_1)}{(n-\ell_2)!\,(n+\ell_2)!} \\ &+ (-1)^{\ell_2} \sum_{n=|\ell_2|}^{|\ell_1|-1} \frac{\xi_1^{2n+1}[1](\ell_1)}{(n-\ell_2)!\,(n+\ell_2)!} \left(\frac{\mathbf{e}_2^+}{n+\ell_2+1} - \frac{\mathbf{e}_2^-}{n-\ell_2+1}\right)\end{aligned} \tag{19}$$

where $\xi_1^{2n+1}[1]$ and $\xi_1^{2n}[1]$ are given by (5)–(6).

Proof

First, consider a point $(\ell_1, 0) \in \mathbb{Z}^2$. The coefficient of δ_0 in f_{2n} is $\binom{2n}{n}$ while its coefficient in f_{2n+1} is $\left(\mathbf{e}_2^+ - \mathbf{e}_2^-\right)\binom{2n+1}{n+1}$. Invoking (10)–(11) we obtain (17). Next, consider a point (ℓ_1, ℓ_2) with $\ell_2 \neq 0$. The coefficient of δ_{ℓ_2} in f_{2n} is given by

$$(-1)^{\ell_2} \frac{(2n)!}{(n-\ell_2)!\,(n+\ell_2)!}$$

while its coefficient in f_{2n+1} is

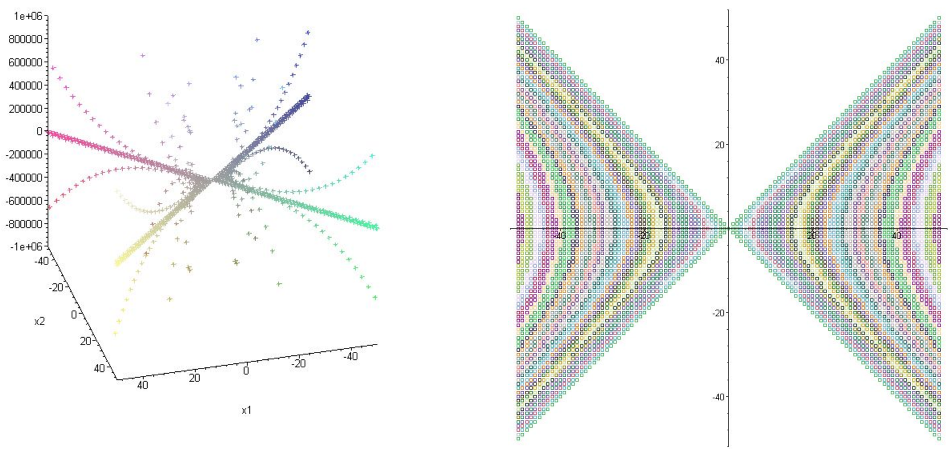
$$\begin{cases} (-1)^{\ell_2-1}\,\mathbf{e}_2^- & \ell_2 = n+1 \\ (-1)^{\ell_2-1} \dfrac{(2n+1)!}{(n+\ell_2)!\,(n-\ell_2+1)!}\,\mathbf{e}_2^- + (-1)^{\ell_2} \dfrac{(2n+1)!}{(n-\ell_2)!\,(n+\ell_2+1)!}\,\mathbf{e}_2^+ & -n \leqslant \ell_2 \leqslant n \\ (-1)^{\ell_2}\,\mathbf{e}_2^+ & \ell_2 = -n-1 \end{cases}$$

Combining this result with (14) we obtain (18)–(19). ■

Corollary 5.1 *The scalar part of $CK[\delta_0]\,(x_1, x_2)$ is given by*

$$\begin{cases} 1 + \displaystyle\sum_{n=1}^{|x_1|} \frac{x_1^2 \prod_{i=1}^{n-1}\left(x_1^2 - i^2\right)}{(n!)^2} & x_2 = 0 \\ (-1)^{x_2} \displaystyle\sum_{n=|x_2|}^{|x_1|} \frac{x_1^2 \prod_{i=1}^{n-1}\left(x_1^2 - i^2\right)}{(n-x_2)!\,(n+x_2)!} & x_2 \neq 0 \end{cases}$$

Remark 5.1 *Figure 1(a) depicts the behaviour of the scalar part of* $CK[\delta_0]$ *between the values* $10^{-6}\ldots10^{6}$ *for* $-50 \leqslant x_1, x_2 \leqslant 50$*, while Figure 1(b) is a contourplot of its absolute value in the same region. At the same time it thus depicts the support in* $\mathbb{Z}^2$ *of* $CK[\delta_0]$*. Note that it follows from Corollary 5.1 that a point* (x_1, x_2) *of the grid* $\mathbb{Z}^2$ *with* $|x_2| > |x_1|$ *cannot belong to the support of the scalar part of* $CK[\delta_0]$*. This specific form of the support is a consequence of the successive actions of* D'*.*



(a) Scalar part of the CK extension

(b) Contourplot

6 Grid with arbitrary mesh width h

In the previous sections the standard case of the grid $\mathbb{Z}^m$ was considered. The aim of this section is to introduce a grid with arbitrary mesh width $h > 0$. In particular, we will investigate how this change of mesh will affect the CK extension of the delta function δ_0. Let $\mathbb{R}^m$ be m-dimensional Euclidean space; over this space a uniform lattice with mesh width $h > 0$ is defined by

$$\mathbb{Z}_h^m = \{(\ell_1 h, \ell_2 h, \ldots, \ell_m h) \mid (\ell_1, \ell_2, \ldots, \ell_m) \in \mathbb{Z}^m\}$$

So a Clifford vector $\underline{x}$ will now only be allowed to show co-ordinates which are integer multiples of the mesh width h. The discrete Dirac operator then depends on the mesh width:

$$D_h[f](\underline{x}) = \sum_{j=1}^{m} \left(\mathbf{e}_j^+ \, \frac{f(\underline{x} + h\mathbf{e}_j) - f(\underline{x})}{h} \; + \; \mathbf{e}_j^- \, \frac{f(\underline{x}) - f(\underline{x} - h\mathbf{e}_j)}{h} \right)$$

as is the case for the star Laplacian:

$$\Delta_h^*[f](\underline{x}) = \sum_{j=1}^{m} \frac{f(\underline{x} + h\mathbf{e}_j) - f(\underline{x} - h\mathbf{e}_j)}{h^2} \; - \; 2m\frac{f(\underline{x})}{h^2}$$

Observe that still $D_h^2 = \Delta_h^*$, and that $D_h \to \partial_{\underline{x}}$ as $h \to 0$ (for the continuous setting with $\mathbf{e}_j^2 = +1$, $j = 1, \ldots, m$, as mentioned before).

The co-ordinate variables are connected with the forward and backward differences through the skew Weyl relations (2)–(3) and thus will also depend on the mesh width h. The action of natural powers of $(\xi_j)_h$ on the ground state 1 is now given by $(\xi_j)_h[1] = x_j\,(\mathbf{e}_j^+ + \mathbf{e}_j^-)$ and

$$\begin{aligned}
(\xi_j)_h^{2n+1}[1] &= x_j\,(\mathbf{e}_j^+ + \mathbf{e}_j^-)\prod_{i=1}^{n}(x_j^2 - i^2h^2) \qquad (20)\\
(\xi_j)_h^{2n}[1] &= \left(x_j^2 + nhx_j\,(\mathbf{e}_j^+\mathbf{e}_j^- - \mathbf{e}_j^-\mathbf{e}_j^+)\right)\prod_{i=1}^{n-1}(x_j^2 - i^2h^2) \qquad (21)
\end{aligned}$$

for $n = 1, 2, \ldots$ and $j = 1, \ldots, m$. As could be expected, we have that $(\xi_j)_h^{2n+1}[1](x_j) \to x_j^{2n+1}\mathbf{e}_j$ and $(\xi_j)_h^{2n}[1](x_j) \to x_j^{2n}$, when $h \to 0$.

Isolating, as before, the first term in the Dirac operator, i.e. writing $D_h = \partial_{1h} + D'_h$, and repeating the calculation of Section 3, we obtain a CK extension which, at least formally, is not affected by the introduction of the mesh width h, though $\xi_1^k[1]$ should be replaced by $(\xi_1)_h^k[1]$.

Definition 6.1 *The CK extension $CK_h\,[f]$ of a discrete function $f(x_2, \ldots, x_m)$ defined on the grid $\mathbb{Z}_h^m$ is the discrete monogenic function*

$$CK_h\,[f]\,(x_1, x_2, \ldots, x_m) = \sum_{k=0}^{\infty} \frac{(\xi_1)_h^k[1]}{k!}\, f_k(x_2, \ldots, x_m)$$

where $f_0 = f$ and $f_{k+1} = (-1)^{k+1} D'_h f_k$.

Seen the behaviour of the natural powers of $(\xi_1)_h$ for $h \to 0$, and the fact that $D'_h \to \partial'_{\underline{x}}$, it directly follows that $\mathrm{CK}_h[f|_{\mathbb{Z}_h^{m-1}}]$ will tend to the CK extension of f in the corresponding continuous setting, reading

$$F(x_1, x_2, \ldots, x_m) = \exp\left(-x_1\mathbf{e}_1\partial_{\underline{x}}{}'\right)[f] = \sum_{k=0}^{\infty} \frac{1}{k!} x_1^k(-\mathbf{e}_1\partial_{\underline{x}}{}')^k[f]$$

for any function f defined on $\mathbb{R}^{m-1}$, which is real-analytic.

For the CK extension of the function δ_0 restricted to the mesh $\mathbb{Z}_h^m$, we will again only consider the case $m = 2$. It is well-defined on the grid $\mathbb{Z}_h^2$ and given by

$$\mathrm{CK}_h(\delta_0)(x_1, x_2) = \sum_{k=0}^{\infty} \frac{(\xi_1)_h^k[1](x_1)}{k!}\, f_k(x_2)$$

with $f_0(x_2) = \delta_0(x_2)$ and $f_{k+1}(x_2) = (-1)^{k+1} D' f_k$, where now

$$\Delta_2^+(\delta_k)(x_2) = \frac{\delta_{k-h}(x_2) - \delta_k(x_2)}{h}, \qquad \Delta_2^-(\delta_k)(x_2) = \frac{\delta_k(x_2) - \delta_{k+h}(x_2)}{h}$$

whence we obtain for the components f_k in $\mathrm{CK}_h[\delta_0]$, $k = 1, 2, \ldots$:

$$f_{2n}(x_2) = \sum_{j=0}^{2n}(-1)^{j+n}\binom{2n}{j}\frac{\delta_{(n-j)h}}{h^{2n}}$$

$$f_{2n+1}(x_2) = \mathbf{e}_2^+\left(\sum_{j=0}^{2n+1}(-1)^{j+n+1}\binom{2n+1}{j}\frac{\delta_{(j-n-1)h}}{h^{2n+1}}\right) + \mathbf{e}_2^-\left(\sum_{j=0}^{2n+1}(-1)^{j+n}\binom{2n+1}{j}\frac{\delta_{(n+1-j)h}}{h^{2n+1}}\right)$$

So $\mathrm{CK}_h[\delta_0]$ can also be rewritten as

$$\mathrm{CK}_h(\delta_0)(x_1,x_2) = \sum_{k=0}^{\infty} \frac{(\xi_1)_h^k[1](x_1)}{h^k\, k!}\, g_k(x_2)$$

where the coefficient functions $g_k(x_2) \equiv h^k f_k(x_2)$ no longer contain any powers of h. We will use this form to evaluate $\mathrm{CK}_h[\delta_0]$ in an arbitrary point $(x_1,x_2) = (\ell_1 h, \ell_2 h)$ of the grid $\mathbb{Z}_h^2$.

Theorem 6.1 *The value of the CK extension of the discrete delta function δ_0 in a point $(\ell_1 h, 0)$ of the grid $\mathbb{Z}_h^2$ is*

$$1 + \sum_{n=1}^{|\ell_1|} \frac{(\xi_1)_h^{2n}[1](\ell_1 h)}{h^{2n}\,(n!)^2} + \sum_{n=0}^{|\ell_1|-1} \frac{(\xi_1)_h^{2n+1}[1](\ell_1 h)}{h^{2n+1}\, n!\,(n+1)!}\,(\mathbf{e}_2^+ - \mathbf{e}_2^-)$$

Furthermore, its value in a point $(\ell_1 h, \ell_2 h)$ with $\ell_2 > 0$ is

$$(-1)^{\ell_2-1}\, \frac{(\xi_1)_h^{2\ell_2-1}[1](\ell_1 h)}{h^{2\ell_2-1}\,(2\ell_2-1)!}\,\mathbf{e}_2^- + (-1)^{\ell_2} \sum_{n=\ell_2}^{|\ell_1|} \frac{(\xi_1)_h^{2n}[1](\ell_1 h)}{h^{2n}\,(n-\ell_2)!\,(n+\ell_2)!}$$
$$+ (-1)^{\ell_2} \sum_{n=\ell_2}^{|\ell_1|-1} \frac{(\xi_1)_h^{2n+1}[1](\ell_1 h)}{h^{2n+1}\,(n-\ell_2)!\,(n+\ell_2)!} \left(\frac{\mathbf{e}_2^+}{n+\ell_2+1} - \frac{\mathbf{e}_2^-}{n-\ell_2+1} \right)$$

while its value in a point $(\ell_1 h, \ell_2 h)$ with $\ell_2 < 0$ is

$$(-1)^{\ell_2}\, \frac{(\xi_1)_h^{2|\ell_2|-1}[1](\ell_1 h)}{h^{2|\ell_2|-1}\,(2\,|\ell_2|-1)!}\,\mathbf{e}_2^+ + (-1)^{\ell_2} \sum_{n=|\ell_2|}^{|\ell_1|} \frac{(\xi_1)_h^{2n}[1](\ell_1 h)}{h^{2n}\,(n-\ell_2)!\,(n+\ell_2)!}$$
$$+ (-1)^{\ell_2} \sum_{n=|\ell_2|}^{|\ell_1|-1} \frac{(\xi_1)_h^{2n+1}[1](\ell_1 h)}{h^{2n+1}\,(n-\ell_2)!\,(n+\ell_2)!} \left(\frac{\mathbf{e}_2^+}{n+\ell_2+1} - \frac{\mathbf{e}_2^-}{n-\ell_2+1} \right)$$

with $(\xi_1)_h^{2n+1}[1]$ and $(\xi_1)_h^{2n}[1]$ given by (20)-(21).

Remark 6.1 *Since δ_0 is defined on $\mathbb{R}^{m-1}$, we may also here consider the limit case for h tending to 0. However, the only powers of h in the above function values appearing as*

$$\frac{\xi_1^{2n+1}}{h^{2n+1}}[1] = \frac{x_1}{h} \prod_{i=1}^{n} \left(\left(\frac{x_1}{h}\right)^2 - i^2 \right) (\mathbf{e}_1^+ + \mathbf{e}_1^-)$$

$$\frac{\xi_j^{2n}[1]}{h^{2n}} = \left(\left(\frac{x_1}{h}\right)^2 + n\frac{x_1}{h}(\mathbf{e}_1^+\mathbf{e}_1^- - \mathbf{e}_1^-\mathbf{e}_1^+) \right) \prod_{i=1}^{n-1} \left(\left(\frac{x_1}{h}\right)^2 - i^2 \right)$$

it is clear that the limit of $CK_h[\delta_0]$ for $h \to 0$ will not exist, since $\frac{x_1}{h} \to \infty$ for a fixed point $(x_1,x_2) \neq (0,0)$. This is in accordance with the fact that δ_0 is not continuous, whence certainly not real-analytic, and thus has no CK extension in the continuous case.

Acknowledgements

The first author acknowledges support by the institutional grant no. B/10675/02 of Ghent University (BOF). The first and second author wish to express their gratitude to Fred Brackx, head of their research group, for fruitful discussions and continuous encouragements.

References

[1] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Research Notes in Mathematics **76**, Pitman, London, 1982.

[2] F. Brackx, H. De Schepper, F. Sommen, L. Van de Voorde, Discrete Clifford analysis: an overview, *Cubo* **11(1)** (2009), 55-71.

[3] F. Brackx, H. De Schepper, F. Sommen, L. Van de Voorde, Discrete Clifford analysis: a germ of function theory. In: I. Sabadini, M. Shapiro, F. Sommen (eds.), *Hypercomplex Analysis*, Birkhäuser, 2009, 37-53.

[4] R. Delanghe, F. Sommen, V. Souček, *Clifford algebra and spinor-valued functions - A function theory for the Dirac operator*, Kluwer Academic Publishers, Dordrecht, 1992.

[5] A. Cauchy, *Oeuvres completes*, Série 1, Tome VII, Gauthier-Villars, Paris, 1882-1974, 17-58.

[6] R. Cooke, The Cauchy–Kovalevskaya Theorem (preprint, available online: http://www.cems.uvm.edu/ cooke/ckthm.pdf).

[7] H. De Ridder, H. De Schepper, F. Sommen, U. Kähler, Discrete function theory based on skew Weyl relations, *Proceedings of the AMS* (to appear).

[8] H. De Ridder, H. De Schepper, F. Sommen, Fueter polynomials in discrete Clifford analysis (submitted).

[9] N. Faustino, U. Kähler, F. Sommen, Discrete Dirac operators in Clifford analysis, *Adv. Appl. Cliff. Alg.* **17(3)** (2007), 451-467.

[10] J. Gilbert, M. Murray, *Clifford Algebra and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge, 1991.

[11] K. Gürlebeck, W. Sprössig, *Quaternionic and Clifford Calculus for Physicists and Engineers*, J. Wiley & Sons, Chichester, 1997.

[12] S. Kowalevsky, Zur Theorie der partiellen Differentialgleichung, *J. für die Reine und Angew. Mathem.* **80**, 1875, 1-32.