Additive generators in interval-valued and intuitionistic fuzzy set theory

Glad Deschrijver
Fuzziness and Uncertainty Modeling Research Unit
Department of Mathematics and Computer Science, Ghent University
Krijgslaan 281 (S9), B-9000 Gent, Belgium
E-mail: Glad.Deschrijver@UGent.be
Homepage: http://www.fuzzy.UGent.be

Abstract

Intuitionistic fuzzy sets in the sense of Atanassov and interval-valued fuzzy sets can be seen as \( \mathcal{L} \)-fuzzy sets w.r.t. a special lattice \( \mathcal{L}' \). Deschrijver [2] introduced additive and multiplicative generators on \( \mathcal{L}' \) based on a special kind of addition introduced in [3]. Actually, many other additions can be introduced. In this paper we investigate additive generators on \( \mathcal{L}' \) as far as possible independently of the addition. For some special additions we investigate which t-norms can be generated by continuous additive generators which are a natural extension of an additive generator on the unit interval.

Keywords: intuitionistic fuzzy set, interval-valued fuzzy set, additive generator, addition on \( \mathcal{L}' \), representable

1 Introduction

Triangular norms on \( ([0,1], \leq) \) were introduced in [18] and play an important role in fuzzy set theory (see e.g. [12, 14] for more details). Generators are very useful in the construction of t-norms: any generator on \( ([0,1], \leq) \) can be used to generate a t-norm. Generators play also an important role in the representation of continuous Archimedean t-norms on \( ([0,1], \leq) \). Moreover, some properties of t-norms which have a generator can be related to properties of their generator. See e.g. [9, 13, 14, 15, 16] for more information about generators on the unit interval.

Interval-valued fuzzy set theory [11, 17] is an extension of fuzzy theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. Another extension of fuzzy set theory is intuitionistic fuzzy set theory introduced by Atanassov [1]. In [6] it is shown that intuitionistic fuzzy set theory is equivalent to interval-valued fuzzy set theory and that both are equivalent to \( \mathcal{L} \)-fuzzy set theory in the sense of Goguen [10] w.r.t. a special lattice \( \mathcal{L}' \). In [2] additive and multiplicative generators on \( \mathcal{L}' \) are investigated based on a special kind of addition introduced in [3]. In [8] another addition was defined. In fact, many more additions can be introduced. Therefore, in this paper we will investigate additive generators on \( \mathcal{L}' \) as far as possible independently of the addition. For some special additions we will investigate which t-norms can be generated
by continuous additive generators which are a natural extension of an additive generator on the unit interval.

2 The lattice \( \mathcal{L}^I \)

Definition 2.1 We define \( \mathcal{L}^I = (L^I, \leq_{L^I}) \), where
\[
L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 \leq x_2\},
\]
\([x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2), \text{ for all } [x_1, x_2], [y_1, y_2] \text{ in } L^I.
\]

Similarly as Lemma 2.1 in [6] it can be shown that \( \mathcal{L}^I \) is a complete lattice.

Definition 2.2 [11, 17] An interval-valued fuzzy set on \( U \) is a mapping \( A : U \to L^I \).

Definition 2.3 [1] An intuitionistic fuzzy set on \( U \) is a set
\[
A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},
\]
where \( \mu_A(u) \in [0, 1] \) denotes the membership degree and \( \nu_A(u) \in [0, 1] \) the non-membership degree of \( u \) in \( A \) and where for all \( u \in U \), \( \mu_A(u) + \nu_A(u) \leq 1 \).

An intuitionistic fuzzy set \( A \) on \( U \) can be represented by the \( \mathcal{L}^I \)-fuzzy set \( A \) given by
\[
A : U \to \mathcal{L}^I : u \mapsto [\mu_A(u), 1 - \nu_A(u)],
\]

In Figure 1 the set \( L^I \) is shown. Note that to each element \( x = [x_1, x_2] \) of \( L^I \) corresponds a point \((x_1, x_2) \in \mathbb{R}^2\).

![Figure 1: The grey area is \( L^I \).](image)

In the sequel, if \( x \in L^I \), then we denote its bounds by \( x_1 \) and \( x_2 \), i.e. \( x = [x_1, x_2] \). The smallest and the largest element of \( \mathcal{L}^I \) are given by \( 0_{\mathcal{L}^I} = [0, 0] \) and \( 1_{\mathcal{L}^I} = [1, 1] \). We define
the relation $\ll_{L^I}$ by $x \ll_{L^I} y \iff (x_1 < y_1$ and $x_2 < y_2)$, for $x, y$ in $L^I$. We define for further usage the sets

$$D = \{[x_1, x_1] \mid x_1 \in [0, 1]\};$$
$$\bar{L}^I = \{(x_1, x_2) \mid (x_1, x_2) \in \mathbb{R}^2$ and $x_1 \leq x_2\};$$
$$\bar{D} = \{[x_1, x_1] \mid x_1 \in \mathbb{R}\};$$
$$\bar{L}^I_+ = \{(x_1, x_2) \mid (x_1, x_2) \in [0, +\infty[^2$ and $x_1 \leq x_2\};$$
$$\bar{D}_+ = \{[x_1, x_1] \mid x_1 \in [0, +\infty]\};$$
$$\bar{L}^I_{\infty,+} = \{(x_1, x_2) \mid (x_1, x_2) \in [0, +\infty[^2$ and $x_1 \leq x_2\};$$
$$\bar{D}_{\infty,+} = \{[x_1, x_1] \mid x_1 \in [0, +\infty]\}.$$

Note that for any non-empty subset $A$ of $L^I$ it holds that

$$\sup A = \sup\{x_1 \mid x_1 \in [0, 1]\} = \sup\{x_2 \mid x_2 \in [0, 1]\} = \sup\{x_1 \mid x_1 \in [0, 2]\} = \sup\{x_2 \mid x_2 \in [0, 2]\}.$$

**Definition 2.4** [4] A t-norm on $L^I$ is a commutative, associative, increasing mapping $T : (L^I)^2 \to L^I$ which satisfies $T(1_L, x) = x$, for all $x \in L^I$. A t-conorm on $L^I$ is a commutative, associative, increasing mapping $S : (L^I)^2 \to L^I$ which satisfies $S(0_L, x) = x$, for all $x \in L^I$.

In [4, 5, 7] the following classes of t-norms on $L^I$ are introduced: let $T$ and $T'$ be t-norms on $([0, 1], \leq)$, then the mappings $T_{T,T'}, T_T, T_{T,T}$ and $T'_T$ given by, for all $x, y$ in $L^I$,

$$T_{T,T'}(x, y) = [T(x_1, y_1), T'(x_2, y_2)], \text{ (t-representable t-norms)}$$
$$T_T(x, y) = [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))], \text{ (pseudo-t-representable t-norms)}$$
$$T_{T,T}(x, y) = [T(x_1, y_1), \max(T(t, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))],$$
$$T'_T(x, y) = [\min(T(x_1, y_2), T(x_2, y_1)), T(x_2, y_2)],$$

are t-norms on $L^I$. The corresponding classes of t-conorms are given by, for all $x, y$ in $L^I$,

$$S_{S,S'}(x, y) = [S(x_1, y_1), S'(x_2, y_2)], \text{ (t-representable t-conorms)}$$
$$S_S(x, y) = [\min(S(x_1, y_2), S(x_2, y_1)), S(x_2, y_2)], \text{ (pseudo-t-representable t-conorms)}$$
$$S_{S,S}(x, y) = [\min(S(1-t, S(x_1, y_1)), S(x_1, y_2), S(x_2, y_1)), S(x_2, y_2)],$$
$$S'_S(x, y) = [S(x_1, y_1), \max(S(x_1, y_2), S(x_2, y_1))].$$

If for a mapping $f$ on $[0, 1]$ and a mapping $F$ on $L^I$ it holds that $F(D) \subseteq D$, and $F([a, a]) = [f(a), f(a)]$, for all $a \in L^I$, then we say that $F$ is a natural extension of $f$ to $L^I$. E.g. $T_{T,T}, T_T, T_{T,T}$ and $T'_T$ are all natural extensions of $T$ to $L^I$.

### 3 Additive generators on $L^I$

In order to investigate additive generators on $L^I$, a suitable addition on $L^I$ is needed. We assume from now on that $\oplus : (L^I \cup L^I_{\infty,+})^2 \to L^I$ satisfies the following natural properties, for all $a, b$ in $L^I \cup L^I_{\infty,+}$,

(i) $\oplus$ is commutative,
(ii) $\oplus$ is associative,

(iii) $\oplus$ is increasing,

(iv) $0_{L^I} \oplus a = a$,

Note that from (iii) and (iv) it follows that $a \oplus b \geq_{L^I} a$, if $b \geq_{L^I} 0_{L^I}$, for all $a, b$ in $L^I \cup L^I_{\infty,+}$.

**Definition 3.1** Let $f : L^I \rightarrow \bar{L}^I_{\infty,+}$ be a strictly decreasing function. The pseudo-inverse $f^{-1} : \bar{L}^I_{\infty,+} \rightarrow L^I$ of $f$ is defined by, for all $y \in \bar{L}^I_{\infty,+}$,

$$f^{-1}(y) = \begin{cases} 
\sup\{x \mid x \in L^I \text{ and } f(x) \gg_{L^I} y\}, & \text{if } y \ll_{L^I} f(0_{L^I}) \\
\sup\{0_{L^I} \cup \{ x \mid x \in L^I \text{ and } (f(x))_1 > y_1 \}
\quad \text{ and } (f(x))_2 \geq (f(0_{L^I}))_2 \}, & \text{if } y_2 \geq (f(0_{L^I}))_2; \\
\sup\{0_{L^I} \cup \{ x \mid x \in L^I \text{ and } (f(x))_2 > y_2 \}
\quad \text{ and } (f(x))_1 \geq (f(0_{L^I}))_1 \}, & \text{if } y_1 \geq (f(0_{L^I}))_1. 
\end{cases}$$

**Definition 3.2** A mapping $f : L^I \rightarrow \bar{L}^I_{\infty,+}$ satisfying the following conditions:

(AG.1) $f$ is strictly decreasing;

(AG.2) $f(1_{L^I}) = 0_{L^I}$;

(AG.3) $f$ is right-continuous in $0_{L^I}$;

(AG.4) $f(x) \oplus f(y) \in R(f)$, for all $x, y \in L^I$, where

$$R(f) = \text{rng}(f) \cup \{ x \mid x \in \bar{L}^I_{\infty,+} \text{ and } [x_1, (f(0_{L^I}))_2] \in \text{rng}(f) \text{ and } x_2 \geq (f(0_{L^I}))_2 \}$$

$$\quad \cup \{ x \mid x \in \bar{L}^I_{\infty,+} \text{ and } [(f(0_{L^I}))_1, x_2] \in \text{rng}(f) \text{ and } x_1 \geq (f(0_{L^I}))_1 \}$$

$$\quad \cup \{ x \mid x \in \bar{L}^I_{\infty,+} \text{ and } x \geq_{L^I} f(0_{L^I}) \};$$

(AG.5) $f^{-1}(f(x)) = x$, for all $x \in L^I$;

is called an additive generator on $L^I$.

**Theorem 3.1** Let $f$ be an additive generator on $([0, 1], \leq)$ and $f : L^I \rightarrow \bar{L}^I_{\infty,+}$ the mapping defined by, for all $x \in L^I$,

$$f(x) = [f(x_2), f(x_1)].$$

Then, for all $y \in \bar{L}^I_{\infty,+}$,

$$f^{-1}(y) = [f^{-1}(y_2), f^{-1}(y_1)].$$

(1)

**Lemma 3.2** Let $f : L^I \rightarrow \bar{L}^I_{\infty,+}$ be a mapping satisfying (AG.1), (AG.2), (AG.3) and (AG.5). Then, for all $x \in L^I$ such that $x_1 > 0$, it holds that $(f(x))_2 < (f(0_{L^I}))_2$ and $(f(x))_1 < (f(0_{L^I}))_1$.

**Lemma 3.3** Let $f : L^I \rightarrow \bar{L}^I_{\infty,+}$ be a mapping satisfying (AG.1), (AG.2), (AG.3) and (AG.5). Then $(f([0, 1]))_1 = (f(0_{L^I}))_1$ or $(f([0, 1]))_2 = (f(0_{L^I}))_2$.

**Corollary 3.4** Let $f : L^I \rightarrow \bar{L}^I_{\infty,+}$ be a mapping satisfying (AG.1), (AG.2), (AG.3), (AG.5) and $f(D) \subseteq D_{\infty,+}$. Then $(f([0, 1]))_2 = (f(0_{L^I}))_2$. 

4
Theorem 3.5 Let \( f : L^I \rightarrow \bar{L}^I_{\infty,+} \) be a continuous mapping satisfying (AG.1), (AG.2), (AG.3), (AG.5) and \( \{D\} \subseteq D_{\infty,+} \). Then there exists an additive generator \( f \) on \([0,1], \leq\) such that, for all \( a \in \bar{L}^I_{\infty,+} \),

\[
f^{(-1)}(a) = [f^{(-1)}(a_2), f^{(-1)}(a_1)].
\]

The following theorem can be shown independently of the addition \( \oplus \) used in (AG.4).

Theorem 3.6 A mapping \( f : L^I \rightarrow \bar{L}^I_{\infty,+} \) is a continuous additive generator on \( L^I \) such that \( f(D) \subseteq D_{\infty,+} \) if and only if there exists a continuous additive generator \( f \) on \([0,1], \leq\) such that, for all \( a \in L^I \),

\[
f(a) = [f(a_2), f(a_1)].
\]

4 Additive generators and t-norms on \( L^I \)

We will give a sufficient condition for \( \oplus \) under which an additive generator associated to \( \oplus \) generates a t-norm. First we give a lemma.

Lemma 4.1 Let \( f \) be an additive generator on \( L^I \) associated to \( \oplus \). If, for all \( x,y,a \) in \( \bar{L}^I_{+} \) such that \( x \leq L^I a \) and \( y \leq L^I a \oplus a \),

\[
y_2 \geq a_2 \implies ((x \oplus y)_1 = (x \oplus [y_1,a_2])_1 \text{ or } \min((x \oplus y)_1, (x \oplus [y_1,a_2])_1) \geq a_1)
\] (3)

and

\[
y_1 \geq a_1 \implies ((x \oplus y)_2 = (x \oplus [a_1,y_2])_2 \text{ or } \min((x \oplus y)_2, (x \oplus [a_1,y_2])_2) \geq a_2)
\] (4)

then, for all \( x \in L^I \) and \( y \in R(f) \), we have that \( f(x) \oplus f(f^{(-1)}(y)) \in R(f) \) and

\[
f^{(-1)}(f(x) \oplus f(f^{(-1)}(y))) = f^{(-1)}(f(x) \oplus y).
\]

Using Lemma 4.1, the following theorem can be shown.

Theorem 4.2 Let \( f \) be an additive generator on \( L^I \) associated to \( \oplus \). If (3) and (4) hold, for all \( x,y,a \) in \( L^I \) such that \( x \leq L^I a \) and \( y \leq L^I a \oplus a \), then the mapping \( T : (L^I)^2 \rightarrow L^I \) defined by, for all \( x,y \) in \( L^I \),

\[
T(x,y) = f^{(-1)}(f(x) \oplus f(y)),
\]

is a t-norm on \( L^I \).

Theorem 4.3 Let \( f \) be an additive generator on \([0,1], \leq\). Then the mapping \( f : L^I \rightarrow \bar{L}^I_{\infty,+} \) defined by, for all \( x \in L^I \),

\[
f(x) = [f(x_2), f(x_1)],
\]

is an additive generator on \( L^I \) associated to \( \oplus \) if and only if, for all \( x,y \) in \( L^I \),

\[
f(x) \oplus f(y) \in (\text{rng}(f) \cup [f(0), +\infty])^2.
\]

Theorem 3.6 and Theorem 3.1 show that no matter which operation \( \oplus \) is used in (AG.4), a continuous additive generator \( f \) on \( L^I \) satisfying \( f(D) \subseteq D_{\infty,+} \) satisfies (2) and its pseudo-inverse satisfies (1). Therefore it depends on the operation \( \oplus \) which classes of t-norms on \( L^I \) can have continuous additive generators that are a natural extension of additive generators on \([0,1], \leq\).
4.1 Additive generators based on $\oplus_{L^I}$

Starting from the observation that the Lukasiewicz t-conorm $S_W$ is given by $S_W(x,y) = \min(1, x + y)$, for all $x, y$ in $[0, 1]$, and that the pseudo-t-representable t-conorm $S_{S_W}$ is given by $S_{S_W}(x,y) = [\min(1, x_1 + y_2, x_2 + y_1), \min(1, x_2 + y_2)]$, for all $x, y$ in $L'$, the following definition of addition on $L'$ is introduced in such a way that $S_{S_W}(x,y) = \inf(1_{L^I}, x \oplus_{L^I} y)$, for all $x, y$ in $L'$.

**Definition 4.1** [3] We define the addition on $L' \cup \bar{L}_\infty^I$ by, for all $x, y$ in $L' \cup \bar{L}_\infty^I$,

$$x \oplus_{L^I} y = [\min(x_1 + y_2, x_2 + y_1), x_2 + y_2],$$

where, for all $x \in \mathbb{R}$, $x + \infty = +\infty$ and $+\infty + \infty = +\infty$.

The other arithmetic operations introduced in [3] allow to write also some other important operations on $L'$, such as the Lukasiewicz t-norm, the product t-norm on $L'$ and their residual implications, using a similar algebraic formula as their counterparts on $([0, 1], \leq)$.

**Theorem 4.4** [2] Let $f$ be any generator on $L'$ associated to $\oplus_{L^I}$. Then the mapping $T : (L')^2 \rightarrow L'$ defined by, for all $x, y$ in $L'$,

$$T(x, y) = f^{-1}(f(x) \oplus_{L^I} f(y)),$$

is a t-norm on $L'$.

**Theorem 4.5** [2] Let $f$ be a continuous additive generator on $L'$ associated to $\oplus_{L^I}$ for which $f(D) \subseteq D_{\infty,+}$. Then there exists a t-norm $T$ on $([0, 1], \leq)$ such that, for all $x, y$ in $L'$,

$$f^{-1}(f(x) \oplus_{L^I} f(y)) = T(x, y).$$

Thus, using $\oplus_{L^I}$, only pseudo-t-representable t-norms on $L'$ can have continuous additive generators $f$ for which $f(D) \subseteq D_{\infty,+}$. Other natural extensions of t-norms on $([0, 1], \leq)$ which have a continuous generator $f$ cannot have a continuous additive generator on $L'$ that is a natural extension of $f$.

4.2 Additive generators based on $\oplus^I_{L^I}$

Now we discuss a second type of addition on $L'$ which was introduced in [8]. Similarly as for $\oplus_{L^I}$, we have that $S_{S_{S_W},t}(x,y) = \inf(1_{L^I}, x \oplus^I_{L^I} y)$, for all $x, y$ in $L'$.

**Definition 4.2** [8] Let $t \in [0, 1]$. Then we define the t-addition on $L' \cup \bar{L}_\infty^I$ by, for all $x, y$ in $L' \cup \bar{L}_\infty^I$,

$$x \oplus^I_{L^I} y = [\min(1 - t + x_1 + y_1, x_1 + y_2, x_2 + y_1), x_2 + y_2].$$

**Theorem 4.6** Let $t \in [0, 1]$ and $f$ be any generator on $L'$ associated to $\oplus^I_{L^I}$. Then the mapping $T : (L')^2 \rightarrow L'$ defined by, for all $x, y$ in $L'$,

$$T(x, y) = f^{-1}(f(x) \oplus^I_{L^I} f(y)),$$

is a t-norm on $L'$.  

6
Theorem 4.7 Let \( t \in [0,1] \) and \( f \) be a continuous additive generator on \( L^I \) associated to \( \oplus_{L^I} \), for which \( f(D) \subseteq D_{\infty,+} \). Then there exists a t-norm \( T \) on \( ([0,1], \leq) \) such that, for all \( x, y \) in \( L^I \),
\[
f^{-1}(f(x) \oplus_{L^I} f(y)) = T_{T,f^{-1}(1-t)}(x,y).
\]

Similarly as for \( \oplus_{L^I} \), from Theorem 4.7 it follows that a t-norm \( T \) on \( L^I \) which is a natural extension of a t-norm on \( ([0,1], \leq) \) generated by a continuous additive generator \( f \) can only have a continuous additive generator associated to \( \oplus_{L^I} \) which is a natural extension of \( f \), if \( T \) belongs to the class of t-norms \( T_{T,f} \).

4.3 Additive generators based on \( \oplus'_{L^I} \)

Finally, we introduce the following addition on \( \bar{L}^I \).

**Definition 4.3** We define the addition on \( \bar{L}^I \cup \bar{L}^I_{\infty,+} \) by, for all \( x, y \) in \( \bar{L}^I \cup \bar{L}^I_{\infty,+} \),
\[
x \oplus'_{L^I} y = [x_1 + y_1, \max(x_1 + y_2, x_2 + y_1)].
\]

This addition is closely related to the t-conorm \( S'_SW \): for all \( x, y \) in \( L^I \), \( S'_SW(x,y) = \inf(1_{L^I}, x \oplus'_{L^I} y) \).

Theorem 4.8 Let \( f \) be any generator on \( L^I \) associated to \( \oplus'_{L^I} \) such that \( f(0_{L^I}) \in D_{\infty,+} \). Then the mapping \( T : (L^I)^2 \rightarrow L^I \) defined by, for all \( x, y \) in \( L^I \),
\[
T(x,y) = f^{-1}(f(x) \oplus'_{L^I} f(y)),
\]
is a t-norm on \( L^I \).

Theorem 4.9 Let \( f \) be a continuous additive generator on \( L^I \) associated to \( \oplus'_{L^I} \), for which \( f(D) \subseteq D_{\infty,+} \). Then there exists a t-norm \( T \) on \( ([0,1], \leq) \) such that, for all \( x, y \) in \( L^I \),
\[
f^{-1}(f(x) \oplus'_{L^I} f(y)) = T_{T,f}^I(x,y).
\]

Similarly as for the two other additions, only t-norms on \( L^I \) belonging to the class of t-norms \( T_{T,f}^I \) can have continuous additive generators \( f \) associated to \( \oplus'_{L^I} \) which are a natural extension of a continuous additive generator on \( ([0,1], \leq) \).

5 Conclusion

In [3, 8] two kinds of arithmetic operations on \( L^I \) are introduced. In [2] one of these kinds of operations is used to construct additive generators on \( L^I \). Since these are not the only possible ways to define addition, subtraction, multiplication and division on \( L^I \), we developed a new theory of additive generators on \( L^I \) as much as possible independently of the addition needed. We found a sufficient condition for \( \oplus \) such that additive generators associated to \( \oplus \) generate t-norms on \( L^I \). We showed that continuous additive generators on \( L^I \) which are a natural extension to \( L^I \) of a generator on \( ([0,1], \leq) \) can be represented in a unique way by the generator on \( ([0,1], \leq) \). As a consequence, the choice of the operation \( \oplus \) determines which classes of t-norms on \( L^I \) can have continuous additive generators which form a natural extension of a generator on the unit interval.
References


