Additive generators in interval-valued and intuitionistic fuzzy set theory

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Abstract

Intuitionistic fuzzy sets in the sense of Atanassov and interval-valued fuzzy sets can be seen as $\mathcal{L}$-fuzzy sets w.r.t. a special lattice $\mathcal{L}^I$. Deschrijver [2] introduced additive and multiplicative generators on $\mathcal{L}^I$ based on a special kind of addition introduced in [3]. Actually, many other additions can be introduced. In this paper we investigate additive generators on $\mathcal{L}^I$ as far as possible independently of the addition. For some special additions we investigate which t-norms can be generated by continuous additive generators which are a natural extension of an additive generator on the unit interval.

Keywords: intuitionistic fuzzy set, interval-valued fuzzy set, additive generator, addition on L^I, representable

1 Introduction

Triangular norms on $([0,1], \leq)$ were introduced in [18] and play an important role in fuzzy set theory (see e.g. [12, 14] for more details). Generators are very useful in the construction of t-norms: any generator on $([0,1], \leq)$ can be used to generate a t-norm. Generators play also an important role in the representation of continuous Archimedean t-norms on $([0,1], \leq)$. Moreover, some properties of t-norms which have a generator can be related to properties of their generator. See e.g. [9, 13, 14, 15, 16] for more information about generators on the unit interval.

Interval-valued fuzzy set theory [11, 17] is an extension of fuzzy theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. Another extension of fuzzy set theory is intuitionistic fuzzy set theory introduced by Atanassov [1]. In [6] it is shown that intuitionistic fuzzy set theory is equivalent to interval-valued fuzzy set theory and that both are equivalent to L-fuzzy set theory in the sense of Goguen [10] w.r.t. a special lattice $\mathcal{L}^I$. In [2] additive and multiplicative generators on $\mathcal{L}^I$ are investigated based on a special kind of addition introduced in [3]. In [8] another addition was defined. In fact, many more additions can be introduced. Therefore, in this paper we will investigate additive generators on $\mathcal{L}^I$ as far as possible independently of the addition. For some special additions we will investigate which t-norms can be generated by continuous additive generators which are a natural extension of an additive generator on the unit interval.

2 The lattice $\mathcal{L}^I$

Definition 2.1 *We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where*

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\},$$
$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2), \text{ for all } [x_1, x_2], [y_1, y_2] \text{ in } L^I.$$

Similarly as Lemma 2.1 in [6] it can be shown that $\mathcal{L}^I$ is a complete lattice.

Definition 2.2 [11, 17] *An interval-valued fuzzy set on U is a mapping $A : U \to L^I$.*

Definition 2.3 [1] *An intuitionistic fuzzy set on U is a set*

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

where $\mu_A(u) \in [0,1]$ denotes the membership degree and $\nu_A(u) \in [0,1]$ the non-membership degree of u in A and where for all $u \in U$, $\mu_A(u) + \nu_A(u) \leq 1$.

An intuitionistic fuzzy set A on U can be represented by the $\mathcal{L}^I$-fuzzy set A given by

$$\begin{aligned} A \,:\, U &\to L^I \,: \\ u &\mapsto [\mu_A(u), 1 - \nu_A(u)], \end{aligned}$$

In Figure 1 the set L^I is shown. Note that to each element $x = [x_1, x_2]$ of L^I corresponds a point $(x_1, x_2) \in \mathbb{R}^2$.

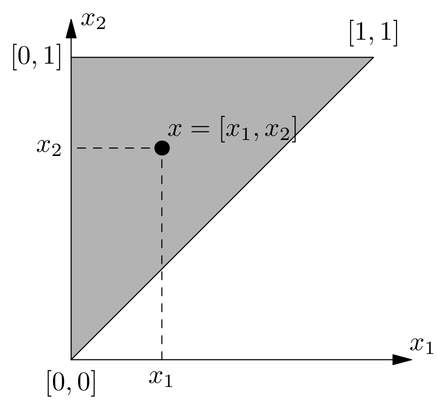

Figure 1: The grey area is L^I.

In the sequel, if $x \in L^I$, then we denote its bounds by x_1 and x_2, i.e. $x = [x_1, x_2]$. The smallest and the largest element of $\mathcal{L}^I$ are given by $0_{\mathcal{L}^I} = [0,0]$ and $1_{\mathcal{L}^I} = [1,1]$. We define the relation $\ll_{L^I}$ by $x \ll_{L^I} y \iff (x_1 < y_1 \text{ and } x_2 < y_2)$, for x, y in L^I. We define for further usage the sets

$$\begin{aligned}
D &= \{[x_1, x_1] \mid x_1 \in [0,1]\}; \\
\bar{L}^I &= \{[x_1, x_2] \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } x_1 \leq x_2\}; \\
\bar{D} &= \{[x_1, x_1] \mid x_1 \in \mathbb{R}\}; \\
\bar{L}^I_+ &= \{[x_1, x_2] \mid (x_1, x_2) \in [0, +\infty[^2 \text{ and } x_1 \leq x_2\}; \\
\bar{D}_+ &= \{[x_1, x_1] \mid x_1 \in [0, +\infty[\}; \\
\bar{L}^I_{\infty,+} &= \{[x_1, x_2] \mid (x_1, x_2) \in [0, +\infty]^2 \text{ and } x_1 \leq x_2\}; \\
\bar{D}_{\infty,+} &= \{[x_1, x_1] \mid x_1 \in [0, +\infty]\}.
\end{aligned}$$

Note that for any non-empty subset A of L^I it holds that

$$\begin{aligned}
\sup A = [&\sup\{x_1 \mid x_1 \in [0,1] \text{ and } (\exists x_2 \in [x_1, 1])([x_1, x_2] \in A)\}, \\
&\sup\{x_2 \mid x_2 \in [0,1] \text{ and } (\exists x_1 \in [0, x_2])([x_1, x_2] \in A)\}].
\end{aligned}$$

Definition 2.4 [4] *A t-norm on $\mathcal{L}^I$ is a commutative, associative, increasing mapping $\mathcal{T} : (L^I)^2 \to L^I$ which satisfies $\mathcal{T}(1_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$. A t-conorm on $\mathcal{L}^I$ is a commutative, associative, increasing mapping $\mathcal{S} : (L^I)^2 \to L^I$ which satisfies $\mathcal{S}(0_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.*

In [4, 5, 7] the following classes of t-norms on $\mathcal{L}^I$ are introduced: let T and T' be t-norms on $([0,1], \leq)$, then the mappings $\mathcal{T}_{T,T'}$, $\mathcal{T}_T$, $\mathcal{T}_{T,t}$ and $\mathcal{T}'_T$ given by, for all x, y in L^I,

$$\begin{aligned}
\mathcal{T}_{T,T'}(x, y) &= [T(x_1, y_1), T'(x_2, y_2)], \text{ (t-representable t-norms)} \\
\mathcal{T}_T(x, y) &= [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))], \text{ (pseudo-t-representable t-norms)} \\
\mathcal{T}_{T,t}(x, y) &= [T(x_1, y_1), \max(T(t, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))], \\
\mathcal{T}'_T(x, y) &= [\min(T(x_1, y_2), T(x_2, y_1)), T(x_2, y_2)],
\end{aligned}$$

are t-norms on $\mathcal{L}^I$. The corresponding classes of t-conorms are given by, for all x, y in L^I,

$$\begin{aligned}
\mathcal{S}_{S,S'}(x, y) &= [S(x_1, y_1), S'(x_2, y_2)], \text{ (t-representable t-conorms)} \\
\mathcal{S}_S(x, y) &= [\min(S(x_1, y_2), S(x_2, y_1)), S(x_2, y_2)], \text{ (pseudo-t-representable t-conorms)} \\
\mathcal{S}_{S,t}(x, y) &= [\min(S(1-t, S(x_1, y_1)), S(x_1, y_2), S(x_2, y_1)), S(x_2, y_2)], \\
\mathcal{S}'_S(x, y) &= [S(x_1, y_1), \max(S(x_1, y_2), S(x_2, y_1))].
\end{aligned}$$

If for a mapping f on $[0,1]$ and a mapping F on L^I it holds that $F(D) \subseteq \bar{D}$, and $F([a, a]) = [f(a), f(a)]$, for all $a \in L^I$, then we say that F is a natural extension of f to L^I. E.g. $\mathcal{T}_{T,T}$, $\mathcal{T}_T$, $\mathcal{T}_{T,t}$ and $\mathcal{T}'_T$ are all natural extensions of T to L^I.

3 Additive generators on $\mathcal{L}^I$

In order to investigate additive generators on $\mathcal{L}^I$, a suitable addition on $\bar{L}^I$ is needed. We assume from now on that $\oplus : (\bar{L}^I \cup \bar{L}^I_{\infty,+})^2 \to \bar{L}^I$ satisfies the following natural properties, for all a, b in $\bar{L}^I \cup \bar{L}^I_{\infty,+}$,

(i) $\oplus$ is commutative,

(ii) $\oplus$ is associative,

(iii) $\oplus$ is increasing,

(iv) $0_{\mathcal{L}^I} \oplus a = a$,

Note that from (iii) and (iv) it follows that $a \oplus b \geq_{L^I} a$, if $b \geq_{L^I} 0_{\mathcal{L}^I}$, for all a, b in $\bar{L}^I \cup \bar{L}^I_{\infty,+}$.

Definition 3.1 *Let $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ be a strictly decreasing function. The pseudo-inverse $\mathfrak{f}^{(-1)} : \bar{L}^I_{\infty,+} \to L^I$ of $\mathfrak{f}$ is defined by, for all $y \in \bar{L}^I_{\infty,+}$,*

$$\mathfrak{f}^{(-1)}(y) = \begin{cases} \sup\{x \mid x \in L^I \text{ and } \mathfrak{f}(x) \gg_{L^I} y\}, & \text{if } y \ll_{L^I} \mathfrak{f}(0_{\mathcal{L}^I}); \\ \sup(\{0_{\mathcal{L}^I}\} \cup \{x \mid x \in L^I \text{ and } (\mathfrak{f}(x))_1 > y_1 \\ \quad \text{and } (\mathfrak{f}(x))_2 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_2\}), & \text{if } y_2 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_2; \\ \sup(\{0_{\mathcal{L}^I}\} \cup \{x \mid x \in L^I \text{ and } (\mathfrak{f}(x))_2 > y_2 \\ \quad \text{and } (\mathfrak{f}(x))_1 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_1\}), & \text{if } y_1 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_1. \end{cases}$$

Definition 3.2 *A mapping $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ satisfying the following conditions:*

(AG.1) *$\mathfrak{f}$ is strictly decreasing;*

(AG.2) $\mathfrak{f}(1_{\mathcal{L}^I}) = 0_{\mathcal{L}^I}$;

(AG.3) *$\mathfrak{f}$ is right-continuous in $0_{\mathcal{L}^I}$;*

(AG.4) *$\mathfrak{f}(x) \oplus \mathfrak{f}(y) \in \mathcal{R}(\mathfrak{f})$, for all x, y in L^I, where*

$$\begin{aligned} \mathcal{R}(\mathfrak{f}) = \operatorname{rng}(\mathfrak{f}) &\cup \{x \mid x \in \bar{L}^I_{\infty,+} \text{ and } [x_1, (\mathfrak{f}(0_{\mathcal{L}^I}))_2] \in \operatorname{rng}(\mathfrak{f}) \text{ and } x_2 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_2\} \\ &\cup \{x \mid x \in \bar{L}^I_{\infty,+} \text{ and } [(\mathfrak{f}(0_{\mathcal{L}^I}))_1, x_2] \in \operatorname{rng}(\mathfrak{f}) \text{ and } x_1 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_1\} \\ &\cup \{x \mid x \in \bar{L}^I_{\infty,+} \text{ and } x \geq_{L^I} \mathfrak{f}(0_{\mathcal{L}^I})\}; \end{aligned}$$

(AG.5) *$\mathfrak{f}^{(-1)}(\mathfrak{f}(x)) = x$, for all $x \in L^I$;*

is called an additive generator on $\mathcal{L}^I$.

Theorem 3.1 *Let f be an additive generator on $([0,1], \leq)$ and $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ the mapping defined by, for all $x \in L^I$,*

$$\mathfrak{f}(x) = [f(x_2), f(x_1)].$$

Then, for all $y \in \bar{L}^I_{\infty,+}$,

$$\mathfrak{f}^{(-1)}(y) = [f^{(-1)}(y_2), f^{(-1)}(y_1)]. \tag{1}$$

Lemma 3.2 *Let $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ be a mapping satisfying (AG.1), (AG.2), (AG.3) and (AG.5). Then, for all $x \in L^I$ such that $x_1 > 0$, it holds that $(\mathfrak{f}(x))_2 < (\mathfrak{f}(0_{\mathcal{L}^I}))_2$ and $(\mathfrak{f}(x))_1 < (\mathfrak{f}(0_{\mathcal{L}^I}))_1$.*

Lemma 3.3 *Let $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ be a mapping satisfying (AG.1), (AG.2), (AG.3) and (AG.5). Then $(\mathfrak{f}([0,1]))_1 = (\mathfrak{f}(0_{\mathcal{L}^I}))_1$ or $(\mathfrak{f}([0,1]))_2 = (\mathfrak{f}(0_{\mathcal{L}^I}))_2$.*

Corollary 3.4 *Let $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ be a mapping satisfying (AG.1), (AG.2), (AG.3), (AG.5) and $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$. Then $(\mathfrak{f}([0,1]))_2 = (\mathfrak{f}(0_{\mathcal{L}^I}))_2$.*

Theorem 3.5 *Let* $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ *be a continuous mapping satisfying (AG.1), (AG.2), (AG.3), (AG.5) and* $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$. *Then there exists an additive generator* f *on* $([0,1], \le)$ *such that, for all* $a \in \bar{L}^I_{\infty,+}$,

$$\mathfrak{f}^{(-1)}(a) = [f^{(-1)}(a_2), f^{(-1)}(a_1)].$$

The following theorem can be shown independently of the addition $\oplus$ used in (AG.4).

Theorem 3.6 *A mapping* $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ *is a continuous additive generator on* $\mathcal{L}^I$ *such that* $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$ *if and only if there exists a continuous additive generator* f *on* $([0,1], \le)$ *such that, for all* $a \in L^I$,

$$\mathfrak{f}(a) = [f(a_2), f(a_1)]. \tag{2}$$

4 Additive generators and t-norms on $\mathcal{L}^I$

We will give a sufficient condition for $\oplus$ under which an additive generator associated to $\oplus$ generates a t-norm. First we give a lemma.

Lemma 4.1 *Let* $\mathfrak{f}$ *be an additive generator on* $\mathcal{L}^I$ *associated to* $\oplus$. *If, for all* x, y, a *in* $\bar{L}^I_+$ *such that* $x \le_{L^I} a$ *and* $y \le_{L^I} a \oplus a$,

$$y_2 \ge a_2 \implies \Big((x \oplus y)_1 = (x \oplus [y_1, a_2])_1 \text{ or } \min((x \oplus y)_1, (x \oplus [y_1, a_2])_1) \ge a_1\Big) \tag{3}$$

and

$$y_1 \ge a_1 \implies \Big((x \oplus y)_2 = (x \oplus [a_1, y_2])_2 \text{ or } \min((x \oplus y)_2, (x \oplus [a_1, y_2])_2) \ge a_2\Big) \tag{4}$$

then, for all $x \in L^I$ *and* $y \in \mathcal{R}(\mathfrak{f})$, *we have that* $\mathfrak{f}(x) \oplus \mathfrak{f}(\mathfrak{f}^{(-1)}(y)) \in \mathcal{R}(\mathfrak{f})$ *and*

$$\mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus \mathfrak{f}(\mathfrak{f}^{(-1)}(y))) = \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus y).$$

Using Lemma 4.1, the following theorem can be shown.

Theorem 4.2 *Let* $\mathfrak{f}$ *be an additive generator on* $\mathcal{L}^I$ *associated to* $\oplus$. *If* (3) *and* (4) *hold, for all* x, y, a *in* L^I *such that* $x \le_{L^I} a$ *and* $y \le_{L^I} a \oplus a$, *then the mapping* $\mathcal{T} : (L^I)^2 \to L^I$ *defined by, for all* x, y *in* L^I,

$$\mathcal{T}(x, y) = \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus \mathfrak{f}(y)),$$

is a t-norm on $\mathcal{L}^I$.

Theorem 4.3 *Let* f *be an additive generator on* $([0,1], \le)$. *Then the mapping* $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ *defined by, for all* $x \in L^I$,

$$\mathfrak{f}(x) = [f(x_2), f(x_1)],$$

is an additive generator on $\mathcal{L}^I$ *associated to* $\oplus$ *if and only if, for all* x, y *in* L^I,

$$\mathfrak{f}(x) \oplus \mathfrak{f}(y) \in (\text{rng}(f) \cup [f(0), +\infty])^2.$$

Theorem 3.6 and Theorem 3.1 show that no matter which operation $\oplus$ is used in (AG.4), a continuous additive generator $\mathfrak{f}$ on $\mathcal{L}^I$ satisfying $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$ satisfies (2) and its pseudo-inverse satisfies (1). Therefore it depends on the operation $\oplus$ which classes of t-norms on $\mathcal{L}^I$ can have continuous additive generators that are a natural extension of additive generators on $([0,1], \le)$.

4.1 Additive generators based on $\oplus_{\mathcal{L}^I}$

Starting from the observation that the Łukasiewicz t-conorm S_W is given by $S_W(x,y) = \min(1, x+y)$, for all x, y in $[0,1]$, and that the pseudo-t-representable t-conorm $\mathcal{S}_{S_W}$ is given by $\mathcal{S}_{S_W}(x,y) = [\min(1, x_1+y_2, x_2+y_1), \min(1, x_2+y_2)]$, for all x, y in L^I, the following definition of addition on $\bar{L}^I$ is introduced in such a way that $\mathcal{S}_{S_W}(x,y) = \inf(1_{\mathcal{L}^I}, x \oplus_{\mathcal{L}^I} y)$, for all x, y in L^I.

Definition 4.1 [3] *We define the addition on $\bar{L}^I \cup \bar{L}^I_{\infty,+}$ by, for all x, y in $\bar{L}^I \cup \bar{L}^I_{\infty,+}$,*

$$x \oplus_{\mathcal{L}^I} y = [\min(x_1+y_2, x_2+y_1), x_2+y_2],$$

where, for all $x \in \mathbb{R}$, $x + \infty = +\infty$ and $+\infty + \infty = +\infty$.

The other arithmetic operations introduced in [3] allow to write also some other important operations on $\mathcal{L}^I$, such as the Łukasiewicz t-norm, the product t-norm on $\mathcal{L}^I$ and their residual implications, using a similar algebraic formula as their counterparts on $([0,1], \leq)$.

Theorem 4.4 [2] *Let $\mathfrak{f}$ be any generator on $\mathcal{L}^I$ associated to $\oplus_{\mathcal{L}^I}$. Then the mapping $\mathcal{T} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathcal{T}(x,y) = \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus_{\mathcal{L}^I} \mathfrak{f}(y)),$$

is a t-norm on $\mathcal{L}^I$.

Theorem 4.5 [2] *Let $\mathfrak{f}$ be a continuous additive generator on $\mathcal{L}^I$ associated to $\oplus_{\mathcal{L}^I}$ for which $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$. Then there exists a t-norm T on $([0,1], \leq)$ such that, for all x, y in L^I,*

$$\mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus_{\mathcal{L}^I} \mathfrak{f}(y)) = \mathcal{T}_T(x,y).$$

Thus, using $\oplus_{\mathcal{L}^I}$, only pseudo-t-representable t-norms on $\mathcal{L}^I$ can have continuous additive generators $\mathfrak{f}$ for which $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$. Other natural extensions of t-norms on $([0,1], \leq)$ which have a continuous generator f cannot have a continuous additive generator on $\mathcal{L}^I$ that is a natural extension of f.

4.2 Additive generators based on $\oplus^t_{\mathcal{L}^I}$

Now we discuss a second type of addition on $\bar{L}^I$ which was introduced in [8]. Similarly as for $\oplus_{\mathcal{L}^I}$, we have that $\mathcal{S}_{S_W,t}(x,y) = \inf(1_{\mathcal{L}^I}, x \oplus^t_{\mathcal{L}^I} y)$, for all x, y in L^I.

Definition 4.2 [8] *Let $t \in [0,1]$. Then we define the t-addition on $\bar{L}^I \cup \bar{L}^I_{\infty,+}$ by, for all x, y in $\bar{L}^I \cup \bar{L}^I_{\infty,+}$,*

$$x \oplus^t_{\mathcal{L}^I} y = [\min(1 - t + x_1 + y_1, x_1+y_2, x_2+y_1), x_2+y_2].$$

Theorem 4.6 *Let $t \in [0,1]$ and $\mathfrak{f}$ be any generator on $\mathcal{L}^I$ associated to $\oplus^t_{\mathcal{L}^I}$. Then the mapping $\mathcal{T} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathcal{T}(x,y) = \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus^t_{\mathcal{L}^I} \mathfrak{f}(y)),$$

is a t-norm on $\mathcal{L}^I$.

Theorem 4.7 *Let $t \in [0,1]$ and $\mathfrak{f}$ be a continuous additive generator on $\mathcal{L}^I$ associated to $\oplus^t_{\mathcal{L}^I}$ for which $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$. Then there exists a t-norm T on $([0,1], \leq)$ such that, for all x, y in L^I,*

$$\mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus^t_{\mathcal{L}^I} \mathfrak{f}(y)) = \mathcal{T}_{T,f^{-1}(1-t)}(x, y).$$

Similarly as for $\oplus_{\mathcal{L}^I}$, from Theorem 4.7 it follows that a t-norm $\mathcal{T}$ on $\mathcal{L}^I$ which is a natural extension of a t-norm on $([0,1], \leq)$ generated by a continuous additive generator f can only have a continuous additive generator associated to $\oplus^t_{\mathcal{L}^I}$ which is a natural extension of f, if $\mathcal{T}$ belongs to the class of t-norms $\mathcal{T}_{T,t}$.

4.3 Additive generators based on $\oplus'_{\mathcal{L}^I}$

Finally, we introduce the following addition on $\bar{L}^I$.

Definition 4.3 *We define the addition on $\bar{L}^I \cup \bar{L}^I_{\infty,+}$ by, for all x, y in $\bar{L}^I \cup \bar{L}^I_{\infty,+}$,*

$$x \oplus'_{\mathcal{L}^I} y = [x_1 + y_1, \max(x_1 + y_2, x_2 + y_1)].$$

This addition is closely related to the t-conorm $\mathcal{S}'_{S_W}$: for all x, y in L^I, $\mathcal{S}'_{S_W}(x, y) = \inf(1_{\mathcal{L}^I}, x \oplus'_{\mathcal{L}^I} y)$.

Theorem 4.8 *Let $\mathfrak{f}$ be any generator on $\mathcal{L}^I$ associated to $\oplus'_{\mathcal{L}^I}$ such that $\mathfrak{f}(0_{\mathcal{L}^I}) \in \bar{D}_{\infty,+}$. Then the mapping $\mathcal{T} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathcal{T}(x, y) = \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus'_{\mathcal{L}^I} \mathfrak{f}(y)),$$

is a t-norm on $\mathcal{L}^I$.

Theorem 4.9 *Let $\mathfrak{f}$ be a continuous additive generator on $\mathcal{L}^I$ associated to $\oplus'_{\mathcal{L}^I}$ for which $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$. Then there exists a t-norm T on $([0,1], \leq)$ such that, for all x, y in L^I,*

$$\mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus'_{\mathcal{L}^I} \mathfrak{f}(y)) = \mathcal{T}'_T(x, y).$$

Similarly as for the two other additions, only t-norms on $\mathcal{L}^I$ belonging to the class of t-norms $\mathcal{T}'_T$ can have continuous additive generators $\mathfrak{f}$ associated to $\oplus'_{\mathcal{L}^I}$ which are a natural extension of a continuous additive generator on $([0,1], \leq)$.

5 Conclusion

In [3, 8] two kinds of arithmetic operations on $\mathcal{L}^I$ are introduced. In [2] one of these kinds of operations is used to construct additive generators on $\mathcal{L}^I$. Since these are not the only possible ways to define addition, substraction, multiplication and division on $\mathcal{L}^I$, we developed a new theory of additive generators on $\mathcal{L}^I$ as much as possible independently of the addition needed. We found a sufficient condition for $\oplus$ such that additive generators associated to $\oplus$ generate t-norms on $\mathcal{L}^I$. We showed that continuous additive generators on $\mathcal{L}^I$ which are a natural extension to $\mathcal{L}^I$ of a generator on $([0,1], \leq)$ can be represented in a unique way by the generator on $([0,1], \leq)$. As a consequence, the choice of the operation $\oplus$ determines which classes of t-norms on $\mathcal{L}^I$ can have continuous additive generators which form a natural extension of a generator on the unit interval.

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