On the algebraic variety $\mathcal{V}_{r,t}$

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Abstract

The variety $\mathcal{V}_{r,t}$ is the image under the Grassmannian map of the $(t-1)$-subspaces of $PG(rt-1, q)$ of the elements of a Desarguesian spread. We investigate some properties of this variety, with particular attention to the case $r = 2$: in this case we prove that every $t + 1$ points of the variety are in general position and we give a new interpretation of linear sets of $PG(1, q')$.

Keywords: Desarguesian spread; Grassmann variety; Veronese variety; Segre variety; subgeometry; linear set.

1 Definitions and preliminary results

Let $V(n, q)$ be the vector space of dimension $n$ over $GF(q)$ and $PG(n-1, q)$ be the projective space defined by the lattice of subspaces of $V(n, q)$; we will denote by $(x_0, \ldots, x_{n-1})$ both the vector of homogeneous coordinates of a certain point $P \in PG(n-1, q)$ and the point $P$ as well. The group $PGL(n, q)$ is the group of all the projectivities of $PG(n-1, q)$. A subspace $\Pi$ of $PG(n-1, q)$ has dimension $t-1$ and rank $t$ if it is a $t$-dimensional subspace of $V(n, q)$. A subgeometry $\Sigma$ of $PG(n-1, q)$ is a subset isomorphic to $PG(n-1, q')$, where $GF(q')$ is a subfield of $GF(q)$. Since a frame consisting of $n+1$ points determines a $PG(n-1, q')$ and $PGL(n, q)$ acts transitively on frames, all the subgeometries $PG(n-1, q')$ contained in $PG(n-1, q)$ are projectively equivalent. It is easy to see that a subgeometry $PG(n-1, q')$ is the set of fixed points of a suitable cyclic semilinear (i.e. $GF(q')$-linear) collineation (see [9], Theorem 4.28 and [6], Chapter 1).

A $(t-1)$-spread $S$ of $PG(n-1, q)$ is a partition of the point set of $PG(n-1, q)$ in subspaces of dimension $(t-1)$ and it exists if and only if $t$ divides $n$ ([16]). Let $S$ be a $(t-1)$-spread of $PG(rt-1, q)$, embed $PG(rt-1, q)$ into $PG(rt, q)$ as a hyperplane and let $A(S)$ be the following incidence structure: the points are the points of $PG(rt, q) \setminus PG(rt-1, q)$, the lines are the $t$-dimensional subspaces of $PG(rt, q)$ intersecting $PG(rt-1, q)$ in an element of $S$ and the incidence is the natural one. Then $A(S)$ is a $2 - (q^{rt}, q^t, 1)$ translation design with parallelism (see [1]) and we will say that $S$ is a Desarguesian spread if $A(S)$ is isomorphic to the affine space $AG(r, q^t)$. An easy construction of a Desarguesian spread of $PG(rt-1, q)$ is by the so called field reduction of $PG(r-1, q^t)$. The underlying vector space of the projective space $PG(r-1, q^t)$ is $V(r, q^t)$; if we consider $V(r, q^t)$ as a vector space over $GF(q)$, then it has dimension $rt$ and it defines a $PG(rt-1, q)$. Every point $P \in PG(r-1, q^t)$ corresponds in this way to a subspace $\Pi_P$ of $PG(rt-1, q)$ of dimension $(t-1)$ and the set $S = \{\Pi_P, P \in PG(r-1, q^t)\}$ is a spread of $PG(rt-1, q)$. Moreover, it is easy to see that any
two elements \( \Pi \) and \( \Pi' \) of \( S \) span a \((2t-1)\)-dimensional subspace completely partitioned by elements of \( S \), and they are precisely the ones corresponding to the points of the line \( \langle P, P' \rangle \) of \( PG(r-1, q^t) \). For \( r > 2 \), such a spread is called normal in [14] and in [1] it is proven that \( S \) is normal if and only if it is Desarguesian; for \( r = 2 \), the proof that a spread constructed in such a way is Desarguesian is in [16].

In [14], a linear set is defined as a generalization of the concept of subgeometry. More precisely, a \( GF(q) \)-linear set \( L \) of \( PG(r-1, q^t) \) of rank \( s \) is a set of points of \( PG(r-1, q^t) \) defined by a subset \( U \) of \( V(r, q^t) \) that is an \( s \)-dimensional vector space over \( GF(q) \). Such a linear set \( L \) is equivalent, by field reduction, to the elements of a Desarguesian spread \( S \) of \( PG(rt-1, q) \) having non-empty intersection with the subspace of \( PG(rt-1, q) \) defined by \( U \). Finally, there is another equivalent way to define a linear set as a (projected) subgeometry of a suitable projective space (for an overview about this topic see [15]). In this paper we present a fourth point of view to describe linear sets of \( PG(1, q^t) \).

We now introduce some algebraic varieties that play an important role in finite geometry.

The Veronese variety \( V(n, d) \) is an algebraic variety of \( PG(\binom{n+d}{d}-1, q) \) image of the injective map \( \nu_{n,d} : PG(n, q) \to PG(\binom{n+d}{d}-1, q) \), where \( \nu_{n,d}(x_0, x_1, \ldots, x_n) \) is the vector of all the monomials of degree \( d \) in \( x_0, \ldots, x_n \) (for \( d = 2 \), see [10], Chapter 25, and for general \( d \) see e.g. [5]) and we recall that \( V(1, d) \) is a normal rational curve of \( PG(d, q) \). We will use the notation \( V(n, d, q) \) to recall also the field under consideration.

Let \( PG(n_1-1, q), PG(n_2-1, q), \ldots, PG(n_k-1, q) \) be \( k \) projective spaces, then the Segre embedding \( \sigma : PG(n_1-1, q) \times PG(n_2-1, q) \times \cdots \times PG(n_k-1, q) \to PG(n_1, n_2, \ldots, n_k-1, q) \) is such that \( \sigma(x^{(1)}_1, x^{(2)}_2, \ldots, x^{(k)}_k) \) is the vector of all the products \( x^{(1)}_j x^{(2)}_j \cdots x^{(k)}_k \), with \( x^i = (x^{(i)}_0, x^{(i)}_1, \ldots, x^{(i)}_{n_i-1}) \in PG(n_i-1, q) \). The image of \( \sigma \) is called the Segre variety \( \Sigma_{n_1,n_2,\ldots,n_k} \) and it is in some way the product of projective spaces (see [10], Chapter 25 and [7]): for this reason we will say the image under \( \sigma \) of the subset \( S_1 \times S_2 \times \cdots \times S_k \) of \( PG(n_1-1, q) \times PG(n_2-1, q) \times \cdots \times PG(n_k-1, q) \) is the Segre product of the subsets \( S_1, S_2, \ldots, S_k \), \( S_i \in PG(n_i-1, q) \). We remark that \( V(n, d) \) is the diagonal of the Segre product of \( d \) \( PG(n, q) \)'s.

To introduce the last variety, we give some more details because the way it is defined is useful in the proof of a proposition of the next section. Let \( \Pi \) be an \((r-1)\)-dimensional subspace of \( PG(n-1, q) \), let \( x^{(1)}, x^{(2)}, \ldots, x^{(r)} \), with \( x^{(i)} \in V(n, q) \) be the coordinate vectors of \( r \) linearly independent points of \( \Pi \) and let \( T_\Pi \) be the matrix whose rows are the vectors \( x^{(1)}, x^{(2)}, \ldots, x^{(r)} \).

After choosing an ordering, we can then construct the vector of length \( \binom{n}{r} \) of all possible \( r \times r \) minors of \( T_\Pi \) and it is called a coordinate vector of \( \Pi \); by Lemma 24.1.1 of [10], this is unique up to a non-zero scalar factor. So we can define the Grassmannian map \( g_{n,r} : PG((r-1)(n-1), q) \to PG((r-1), q) \), where \( PG((r-1)(n-1), q) \) is the set of all \((r-1)\)-subspaces of \( PG(n-1, q) \), such that \( g_{n,r}(\Pi) \) is a coordinate vector of \( \Pi \). This map is injective and its image \( G_{n,r} \Pi \) is called the Grassmannian or the Grassmann variety of the \((r-1)\)-subspaces of \( PG(n-1, q) \) (for more details we refer to [10], Chapter 24).

The varieties described in this section are the image of injective maps, so every collineation of the projective space where the map is defined induces a collineation fixing the variety setwise and viceversa (for the Grassmann and
the Segre variety, see [10] Theorem 24.2.16 and Theorem 25.5.13 respectively; for the Veronese variety, see [5] Theorem 2.15). If σ is a collineation of the projective space, we will denote by σ* the collineation induced on the variety and we will call it the lifting of σ.

2 The algebraic variety $\mathcal{V}_{r,t}$

The algebraic variety $\mathcal{V}_{r,t}$ appeared for the first time in the literature in [16] and it has been described in a more detailed way and with a modern terminology in [14]. This variety is the image under the Grassmannian map $g_{r,t,t}$ of the elements of a Desarguesian $(t-1)$–spread $S$ of $PG(rt-1,q)$; in [14], Lunardon proves that $\mathcal{V}_{r,t}$ is the complete intersection of the Grassmann variety $\mathcal{G}_{r,t,t}$ with a suitable $(t-1)$–space. In fact he proves that $\mathcal{V}_{r,t} = \Delta \cap \Sigma_{r,r_1,...,r_t}$, where $\Delta = PG(r^t-1,q)$ and $\Sigma_{r,r_1,...,r_t}$ is the Segre variety product of $t \: PG(r-1,q^t)$’s contained in the Grassmannian of the $(r-1)$–subspaces of $PG(n-1,q^t)$. As showed in the previous section, by field reduction, we can get a Desarguesian $(t-1)$–spread $S$ of $PG(rt-1,q)$ from $PG(r-1,q^t)$: in this way, to every point $P$ of $PG(r-1,q^t)$ corresponds a spread element $\Pi_P$ and to every line $m$ of $PG(r-1,q^t)$ is isomorphic to $PG(r-1,q^t)$. There are remarkable examples of such varieties: for $r = t = 2$, $\mathcal{V}_{2,2}$ is an elliptic quadric contained in the Klein quadric $Q^+(5,q)$ (see [8], Chapter 16); for $t = 2$, we have the so called Hermitian Veronesean (see for example [4]); for $t = 3, r = 2$ and $q$ even, $\mathcal{V}_{3,2}$ is the Desarguesian ovoid of $Q^+(7,q)$ and for $t = 2, r = 3$ and $q = 2$ mod 3, a suitable hyperplane section of $\mathcal{V}_{2,3}$ is the Unitary ovoid of $Q^+(7,q)$, (see [11, 14]).

We start giving an explicit description of $\mathcal{V}_{r,t}$ in terms of coordinates.

**Proposition 1.** The algebraic variety $\mathcal{V}_{r,t}$ is isomorphic to the set of points of $PG(r^t-1,q^t)$ with coordinates $(x^{a_1}, x^{a_2}, \ldots, x^{a_{r^t}})$, where $x^{a_i} = x_0^{a_{i,0}} x_1^{a_{i,1}} \cdots x_{r^t-1}^{a_{i,r^t-1}}$, $(a_{i,0}, a_{i,1}, \ldots, a_{i,r^t-1})$ is such that $a_{i,k}$ is a sum of distinct powers of $q$, $\sum_{k=0}^{q^{t-1} + q^{t-2} + \ldots + 1} a_{i,k} = q^{t-1}$ and $(x_0, x_1, \ldots, x_{r^t-1}) \in PG(r-1,q^t)$ and it is contained in a subgeometry isomorphic to $PG(r^t-1,q)$.

**Proof.** In $\Sigma^* = PG(rt-1,q^t)$, consider the subgeometry $\Sigma = \{ (x_0, x_1, x_{r-1}, x_0^q, x_1^q, x_{r-1}^q, x_0^{q^2}, x_1^{q^2}, x_{r-1}^{q^2}, \ldots, x_0^{q^{t-1}}, x_1^{q^{t-1}}, x_{r-1}^{q^{t-1}}) : x_i \in GF(q^t) \}$; $\Sigma$ is the set of fixed points of the $GF(q)$–linear collineation

\[ \sigma : (x^{(1)}, x^{(2)}, \ldots, x^{(t)}) \mapsto (x^{(t)q}, x^{(t)q}, \ldots, x^{(t-1)q}), x^{(i)} = (x_0^{(i)}, x_1^{(i)}, x_{r-1}^{(i)}) \in V(r, q^t) \]

of order $t$, hence $\Sigma = PG(tr-1,q)$. Let $\Pi = \{ (x, 0, \ldots, 0) : x \in V(r, q^t) \} \subset \Sigma^*$ and for any $P \in \Pi$ let $\ell(P) = (P, P^q, \ldots, P^{q^{t-1}})$, then $S = \{ \ell(P) : P \in \Pi \}$ is a Desarguesian spread of $\Sigma$ (see [3]). Let $g_{r,t,t}^*$ be the Grassmannian map of subspaces of rank $t$ of $\Sigma^*$: by [14], page 250, the image under $g_{r,t,t}^*$ of the subspaces of rank $t$ of $\Sigma$ is the Grassmannian of $(t-1)$–subspaces of $\Sigma$. The image under $g_{r,t,t}^*$ of $\ell(P)$ is the vector of all minors of order $t$ of the matrix
whose rows are the coordinate vectors of $P,P^r,\ldots,P^{r^{l-1}}$, that is the matrix

$$T(P) = \begin{pmatrix}
x & 0 & \cdots & 0 \\
0 & x^q & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & x^{r^{l-1}}
\end{pmatrix},$$

where $x = (x_0,\ldots,x_{r-1}) \in V(r,q^t)$ and $P = (x,0,\ldots,0) \in \Pi$. The submatrices of order $t$ of $T(P)$ are such that every column has only one non–zero entry, hence the determinant is 0 or it is in the form $x_0^{a(0)} \cdot x_1^{a(1)} \cdots x_{r-1}^{a(r-1)}$, with $\sum_{k=0}^{R} a(k) = q^{t-1}+q^{t-2}+\cdots+1$, $a(k)$ is a sum of distinct powers of $q$. This set of points is contained in a subgeometry isomorphic to $PG(r-1,q)$ by [14], page 250.

**Remark 1** We want to emphasize the analogy of $V_{r,t}$ with the Veronese variety $V(r-1,t,q^t)$. We have already mentioned that $V_{r,t}$ is the intersection of $\Sigma_{r,t,...}$ the Segre variety product of $t \, PG(r-1,q^t)'$s with a suitable subgeometry $PG(r^t-1,q)$, more precisely, it is the Segre embedding of the points of type $(x,x^q,\ldots,x^{q^{t-1}}) \in PG(r-1,q^t) \times PG(r-1,q^t) \times \cdots PG(r-1,q^t)$, whereas $V(r-1,t,q^t)$ is the diagonal of $\Sigma_{r,t,...}$, i.e. is the Segre embedding of the points of type $(x,x,\ldots,x) \in PG(r-1,q^t) \times PG(r-1,q^t) \times \cdots PG(r-1,q^t)$. Moreover, $V(r-1,t,q^t)$ is defined by the vectors of all monomials of degree $t$ in $x_0,x_1,\ldots,x_{r-1}$, whereas $V_{r,t}$ is defined by the vectors of all monomials of degree $1+q+\cdots+q^{t-1}$, but the only powers admitted for $x_i$ are of type $q^{a_1}+\cdots+q^{a_k}$, $a_i \neq a_j \forall i \neq j$.

**Example 1** The variety $V_{3,2}$ is the image of the map $\alpha : (x_0,x_1,x_2) \in PG(2,q^3) \rightarrow (x_0^{q+1},x_0,x_1,x_1^{q+1},x_1x_2,x_2,x_2^{q+1},x_2x_0,x_0) \in PG(8,q^2)$. Let $\sigma$ be the following $GF(q)$–linear collineation of order two:

$$(y_0,y_1,y_2,y_3,y_4,y_5,y_6,y_7) \in PG(8,q^2) \rightarrow (y_0^q,y_1^q,y_2^q,y_3^q,y_4,y_5,y_6,y_7^q) \in PG(8,q^2).$$

The points of $V_{3,2}$ are fixed by $\sigma$ and hence $V_{3,2}$ is contained in the $PG(8,q)$ defined by $\sigma$ (compare with [4]).

**Example 2** The variety $V_{2,4}$ is the image of the map $\alpha : (x,y) \in PG(1,q^4) \rightarrow (x^{q^2+q^3+q},x^{q^3+q^4+q},x^{q^2+q+q^4+q},x^{q^3+q+q^4+q},x^{q^3+q^2+q},x^{q+q^2+q^4+q},x^{q^3+q+q^4+q},x^{q^3+q^2+q},x^{q^2+q^3+q^4+q},x^{q+q^2+q^4+q}) \in PG(15,q^4)$. Let $\tau$ be the following $GF(q)$–linear collineation of order four: $(z_0,z_1,\ldots,z_{15}) \in PG(15,q^4) \rightarrow (z_0^q,z_1^q,z_2^q,z_3^q,z_4^q,z_5^q,z_6^q,z_7^q,z_8^q,z_9^q,z_{10}^q,z_{11}^q,z_{12}^q,z_{13}^q,z_{14}^q,z_{15}^q)$. The points of $V_{2,4}$ are fixed by $\tau$ and hence $V_{2,4}$ is contained in the $PG(15,q)$ defined by $\tau$.

**Remark 2** There is a group isomorphic to $PGL(r,q^t)$ acting 2–transitively on $V_{r,t}$ ([14], Corollary 1).

The following result is the group isomorphic to $PGL(r,q^t)$ acting 2–transitively on $V_{r,t}$ ([14], Corollary 1).

**Theorem 2.** Let $g$ be the map $P \in PG(r-1,q^t) \rightarrow g_{r,t}(\ell(P))$. The image under $g$ of a subgeometry $PG(r-1,q^t)$ is the intersection of the Segre product of $s$ Veronese varieties $V(r-1,\frac{t}{2},q^s)$ with a $PG((r-1+\frac{t}{2})^s-1,q)$ and
it is the complete intersection of $\mathcal{V}_{r,t}$ with a suitable space of rank $(t-\frac{r}{2})^s$.

In particular, the image of a subgeometry $\text{PG}(r-1,q)$ is a Veronese variety $\mathcal{V}(r-1,t,q)$ and it is the intersection of $\mathcal{V}_{r,t}$ with a suitable space of rank $(t^s)^s$.

Proof. Since all the subgeometries are projectively equivalent and by Remark 2, we can assume that the points of $\text{PG}(r-1,q)$ are the ones with coordinates in $\text{GF}(q^s)$. If $P \in \text{PG}(r-1,q^s)$, then the image under the Grassmannian map of $\ell(P)$ is the vector of all minors of order $t$ of the matrix

$$T(P) = \begin{pmatrix} x & 0 & \ldots & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ 0 & x^q & \ldots & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x^{q-1} & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & \ldots & x & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & \ldots & 0 & x^q & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & \ldots & 0 & 0 & \ldots & x^{q-1} \end{pmatrix}$$

where $x = (x_0, \ldots, x_{r-1}) \in V(r,q^s)$. Next, consider the following matrix:

$$T(P)^* = \begin{pmatrix} x_1 & 0 & \ldots & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_s & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & \ldots & x_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \ldots & \ldots & 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & \ldots & 0 & 0 & \ldots & x_s \end{pmatrix}$$

where $x = (x_0, \ldots, x_{r-1}) \in V(r,q^s)$; the vectors of all the minors of $T(P)^*$ is the Segre product of $s$ Veronese varieties $\mathcal{V}(r-1, \frac{r}{2}, q^s)$ and the minors of $T(P)$ are the points of this variety fixed by the $GF(q^s)$–linear collineation $\sigma^s$. Hence, as in [14] page 250, this variety is $V(r-1, \frac{r}{2}, q^s) \cap \Delta$, where $\Delta = \text{PG}(t^{\frac{r}{2}} - 1, q)$.\hfill\Box

2.1 The case $r = 2$

In this section, we focus on the case $r = 2$. In [14], Theorem 1, Lunardon proves that the algebraic variety $\mathcal{V}_{r,t}$ is a cap of $\text{PG}(t^2 - 1, q)$, i.e. any three points of $\mathcal{V}_{r,t}$ are not collinear. In the case $r = 2$, we can prove a stronger result, but we first need a technical lemma.
Lemma 1. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of $n$ distinct non-negative integers, with $n \leq t$ and $\alpha_i < t \ \forall i$. Let $M$ be the $(n+1) \times 2^n$ matrix over $GF(q^t)$, such that the columns of $M$ are in bijective correspondence with the elements of the power set of $S$, namely $\mathcal{P}(S)$, and $M_{i,j} = x_{(j)}^{\alpha_i}$, where $v(j) = q^{\alpha_1} + \ldots + q^{\alpha_i}$ and $\{i_1, \ldots, i_k\}$ is the $j$-th element of $\mathcal{P}(S)$ (by convention, if the $j$-th element is the empty set, then $x_j^{(j)} = 1$). If $x_h \neq x_k \ \forall h \neq k$, then the $GF(q^t)$-rank of $M$ is $n+1$.

Proof. We prove the statement by induction on $n$. For $n = 1$, $M = \begin{pmatrix} 1 & x_1^{q^t} \\ 1 & y_1^{q^t} \end{pmatrix}$ and the statement is obviously true. Let now $n > 1$ and suppose it is true for $n-1$. We assume that the first column is the all-one column. After adding to every column a suitable linear combination of the other ones, we can get a matrix $M'$ such that the first row is the vector $(1,0,\ldots,0)$ and $M'_{i,j} = (x_i - x_j)^{(j)}$, $\forall i = 2, \ldots, n+1$ and $\forall j = 1, \ldots, 2^n$. Consider the submatrix of components $M'_{i,j}$ with $i \geq 2$ and $j$ such that the $j$-th element of $\mathcal{P}(S)$ contains $\alpha_1$; under the hypothesis that $x_i \neq x_1 \ \forall i \geq 2$, we can divide each row by $(x_i - x_1)^{\alpha_1}$ and in this way we get a $n \times 2^{n-1}$ matrix over $GF(q^t)$ determined by the set $S' = S \setminus \{\alpha_1\}$: by the induction hypothesis the rank of this matrix is $n$ and so the rank of $M$ is $n+1$.

Theorem 3. Any $t+1$ points of $\mathcal{V}_{2,t}$ are in general position, i.e. any $t+1$ points of $\mathcal{V}_{2,t}$ span a $t$-dimensional space.

Proof. The points of $\mathcal{V}_{2,t}$ are $\{(x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_k}) \alpha_i \text{ are all the sums of distinct powers } q^i, 0 \leq i \leq t-1 \} \cup \{ P = (0,0,\ldots,0,1) \}$. Since by Remark 2 there is a transitive group fixing $\mathcal{V}_{2,t}$, we can assume that the $t+1$ points we consider are distinct from $P$. Let $M$ be the matrix the rows of which are the coordinate vectors of $t+1$ points of $\mathcal{V}_{2,t} \setminus \{P\}$. We can apply the Lemma 1 to $M$ with $n = t$, hence the $t+1$ rows vectors of $M$ are $GF(q^t)$-linearly independent and so they are also $GF(q)$-linearly independent.

Remark 3 This is another analogy with the Veronese variety: $\mathcal{V}(1,t)$ is a normal rational curve and it has the property that any $t+1$ points span a $t$-dimensional space.

The next theorem is about linear sets of $PG(1,q^t)$. In Section 1 we have recalled the three different ways to define a linear set of a projective geometry, but for our proof we shall use the following: a linear set of $PG(1,q^t)$ of rank $r$ is the set of the elements of $S$, where $S$ is a Desarguesian $(t-1)$-spread of $PG(2t-1,q)$, with non-empty intersection with a subspace of $PG(2t-1,q)$ of dimension $r-1$; in this case, a linear set is a proper one when $r \leq t$.

We need to recall the following property of the Grassmannian. Let $\mathcal{G}$ be the Grassmannian of the $(t-1)$-subspaces of $PG(2t-1,q)$: $\mathcal{G}$ is in $PG(N-1,q)$, where $N = \binom{2t}{2}$. By [10], page 109, in $PG(N-1,q)$ there exists a polarity, called the fundamental polarity of $\mathcal{G}$, such that for every $(t-1)$-space $\Pi$, the $(t-1)$-spaces with non-empty intersection with $\Pi$ correspond to the points of $\mathcal{G} \cap g(\Pi)^\perp$, where $g$ is the Grassmannian map.

Theorem 4. A linear set $L$ of rank $r \leq t$ of $PG(1,q^t)$ corresponds to the points of $\Pi \cap \mathcal{V}_{2,t}$, where $\Pi$ is a suitable subspace of the $PG(2t-1,q)$ containing $\mathcal{V}_{2,t}$. Moreover, if $r = t$, then $\Pi$ is a hyperplane of $PG(2t-1,q)$; if $r = t-1$, then $\Pi$ is a subspace of codimension $t+1$ of $PG(2t-1,q)$.
The points of $L$ correspond to the elements of $S$ intersecting an $(r - 1)$-dimensional subspaces $\Omega$ of $PG(2t - 1, q)$. An element $\pi \in S$ intersects $\Omega$ if and only if $\pi$ intersects all the $(t - 1)$-spaces through $\Omega$. In $PG(N - 1, q)$, let $\Lambda$ be the $(2^t - 1)$-dimensional subspace containing $\mathcal{V}_{2,t}$, and let $\mathcal{G}' = \{g(\pi), \Omega \subseteq \pi\}$: by [10], Corollary 1 page 117, $\mathcal{G}'$ is projectively equivalent to the Grassmannian of the $(t - r - 1)$-spaces of $PG(2t - r - 1, q)$, hence $\langle \mathcal{G}' \rangle = \Sigma$ a $(t^2 - r - 1)$-space. Hence, the points of $L$ correspond to the points of $\mathcal{V}_{2,t} \cap \Sigma$ if $r = t$, then $\Sigma$ is a point and $\mathcal{V}_{2,t} \cap \Sigma^⊥$ is a hyperplane section of $\mathcal{V}_{2,t}$ (since $\mathcal{V}_{2,t}$ can not be contained in the hyperplane because not all the elements of $S$ can intersect a given $(t - 1)$-space). If $r = t - 1$, then $\mathcal{G}'$ is a maximal subspace of $\mathcal{G}$ and it has dimension $t$. The space $\Lambda^⊥$ has empty intersection with $\mathcal{G}$, since no $(t - 1)$-space can intersect all the spread elements, hence $\Lambda^⊥ \cap \mathcal{G}' = \emptyset$, and so $\Lambda \cap \mathcal{G}'^⊥$ is the minimum possible, i.e. it is a subspace of codimension $t + 1$ of $\Lambda$. 

The following result is a generalization of the main result of Section 3 of [12], where Lavrauw and Van de Voorde show how a $GF(q)$-linear set of $PG(1, q^s)$ can intersect a subline $PG(1, q)$. 

**Proposition 5.** A $GF(q)$-linear set $L$ of $PG(1, q^s)$ either contains a fixed subline $PG(1, q^r)$, $s | r$, or it intersects it in at most $\frac{1}{2}(q^{s-1} + q^{s-2} + \ldots + 1)$ points.

**Proof.** The points of $L$ correspond to the points of the intersection of $\mathcal{V}_{2,t}$ with a suitable subspace. The variety $\mathcal{V}_{2,t}$ consists of the points $(x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_m}) \in PG((1 + \frac{1}{t})^s - 1, q)$, where $x^{\alpha_1} = x_0^{\alpha_{(i)}} x_1^{\alpha_{(i)}}$, $(\alpha_{(i)}, \alpha_{(i)})$ is such that $\alpha_{(i)}^{(i)}$ is a sum of distinct powers of $q$, $\alpha_0^{(i)} + \alpha_1^{(i)} = \frac{1}{t}(q^{s-1} + q^{s-2} + \ldots + 1)$ $\forall i$, $x^{\alpha_i} \neq (x^{\alpha_i})y$ $\forall i \neq j$, $\forall h = 0, \ldots, t - 1$, and $(x_0, x_1) \in PG(1, q^s)$. Hence, if a hyperplane section of $\mathcal{V}_{2,t}$ does not contain the image of $PG(1, q^s)$, then it consists of the points corresponding to the points of $PG(1, q^s)$ that satisfy a homogeneous equation of degree $\frac{1}{t}(q^{s-1} + q^{s-2} + \ldots + 1)$ and so they are at most $\frac{1}{t}(q^{s-1} + q^{s-2} + \ldots + 1)$. 

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**References**


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