A Cauchy-Kowalevski theorem for inframonogenic functions

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Abstract

In this paper we prove a Cauchy-Kowalevski theorem for the functions satisfying the system \( \partial_x f \partial_x = 0 \) (called inframonogenic functions).

Keywords: Inframonogenic functions; Cauchy-Kowalevski theorem.

Mathematics Subject Classification: 30G35.

1 Introduction

Let \( \mathbb{R}_{0,m} \) be the \( 2^m \)-dimensional real Clifford algebra constructed over the orthonormal basis \( (e_1, \ldots, e_m) \) of the Euclidean space \( \mathbb{R}^m \) (see [3]). The multiplication in \( \mathbb{R}_{0,m} \) is determined by the relations \( e_j e_k + e_k e_j = -2 \delta_{jk}, \) \( j, k = 1, \ldots, m, \) where \( \delta_{jk} \) is the Kronecker delta. A general element of \( \mathbb{R}_{0,m} \) is of the form

\[
a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},
\]

*accepted for publication in Mathematical Journal of Okayama University
where for $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\}$, $j_1 < \cdots < j_k$, $e_A = e_{j_1} \cdots e_{j_k}$. For the empty set $\emptyset$, we put $e_\emptyset = 1$, the latter being the identity element.

Notice that any $a \in \mathbb{R}_{0,m}$ may also be written as $a = \sum_{k=0}^{m}[a]_k$ where $[a]_k$ is the projection of $a$ on $\mathbb{R}_0^{(k)}$. Here $\mathbb{R}_0^{(k)}$ denotes the subspace of $k$-vectors defined by

$$\mathbb{R}_0^{(k)} = \left\{ a \in \mathbb{R}_{0,m} : a = \sum_{|\lambda|=k} a_\lambda e_\lambda, \ a_\lambda \in \mathbb{R} \right\}.$$

Observe that $\mathbb{R}^{m+1}$ may be naturally identified with $\mathbb{R}_0^{(0)} \oplus \mathbb{R}_0^{(1)}$ by associating to any element $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ the “paravector” $\bar{x} = x_0 + \bar{x} = x_0 + \sum_{j=1}^{m} x_j e_j$.

Conjugation in $\mathbb{R}_{0,m}$ is given by

$$\bar{a} = \sum_A a_A \bar{e}_A,$$

where $\bar{e}_A = \bar{e}_{j_1} \cdots \bar{e}_{j_k}$, $\bar{e}_j = -e_j$, $j = 1, \ldots, m$. One easily checks that $\bar{ab} = \bar{b}\bar{a}$ for any $a, b \in \mathbb{R}_{0,m}$. Moreover, by means of the conjugation a norm $|a|$ may be defined for each $a \in \mathbb{R}_{0,m}$ by putting

$$|a|^2 = [a\bar{a}]_0 = \sum_A a_A^2.$$

Let us denote by $\partial_x = \partial_{x_0} + \partial_{\bar{x}} = \partial_{x_0} + \sum_{j=1}^{m} e_j \partial x_j$ the generalized Cauchy-Riemann operator and let $\Omega$ be an open set of $\mathbb{R}^{m+1}$. According to [11], an $\mathbb{R}_{0,m}$-valued function $f \in C^2(\Omega)$ is called an inframonogenic function in $\Omega$ if and only if it fulfills in $\Omega$ the “sandwich” equation $\partial_x f \partial_x f = 0$.

It is obvious that monogenic functions (i.e. null-solutions of $\partial_x$) are inframonogenic. At this point it is worth remarking that the monogenic functions are the central object of study in Clifford analysis (see [2, 4, 5, 7, 8, 9, 10, 14]). Furthermore, the concept of monogenicity of a function may be seen as the higher dimensional counterpart of holomorphy in the complex plane.

Moreover, as

$$\Delta_x = \sum_{j=0}^{m} \partial^2_{x_j} = \partial_x \partial_x = \partial_x \partial_x,$$

every inframonogenic function $f \in C^4(\Omega)$ satisfies in $\Omega$ the biharmonic equation $\Delta_x^2 f = 0$ (see e.g. [1, 6, 12, 15]).

This paper is intended to study the following Cauchy-type problem for the inframonogenic functions. Given the functions $A_0(\bar{x})$ and $A_1(\bar{x})$ analytic
in an open and connected set \( \Omega \subset \mathbb{R}^m \), find a function \( F(x) \) inframonogenic in some open neighbourhood \( \tilde{\Omega} \) of \( \tilde{\Omega} \) in \( \mathbb{R}^{m+1} \) which satisfies

\[
F(x)|_{x_0=0} = A_0(\underline{x}), \quad (1)
\]
\[
\partial_{x_0} F(x)|_{x_0=0} = A_1(\underline{x}). \quad (2)
\]

2 Cauchy-type problem for inframonogenic functions

Consider the formal series

\[
F(x) = \sum_{n=0}^{\infty} x_0^n A_n(\underline{x}). \quad (3)
\]

It is clear that \( F \) satisfies conditions (1) and (2). We also see at once that

\[
\partial_x (x_0^n A_n) \partial_x - n(n-1)x_0^{n-2} A_n + nx_0^{n-1} (\partial_\underline{x} A_n + A_n \partial_\underline{x}) + x_0^n \partial_\underline{x} A_n \partial_\underline{x}.
\]

We thus get

\[
\partial_x F \partial_x = \sum_{n=0}^{\infty} x_0^n \left( (n+2)(n+1) A_{n+2} + (n+1)(\partial_\underline{x} A_{n+1} + A_{n+1} \partial_\underline{x}) + \partial_\underline{x} A_n \partial_\underline{x} \right).
\]

From the above it follows that \( F \) is inframonogenic if and only if the functions \( A_n \) satisfy the recurrence relation

\[
A_{n+2} = -\frac{1}{(n+2)(n+1)} \left( (n+1)(\partial_\underline{x} A_{n+1} + A_{n+1} \partial_\underline{x}) + \partial_\underline{x} A_n \partial_\underline{x} \right), \quad n \geq 0.
\]

It may be easily proved by induction that

\[
A_n = \frac{(-1)^{n+1}}{n!} \left( \sum_{j=0}^{n-2} \partial_\underline{x}^{n-j-1} A_0 \partial_\underline{x}^{j+1} + \sum_{j=0}^{n-1} \partial_\underline{x}^{n-j-1} A_1 \partial_\underline{x}^j \right), \quad n \geq 2. \quad (4)
\]

We now proceed to examine the convergence of the series (3) with the functions \( A_n \) \((n \geq 2)\) given by (4). Let \( \underline{y} \) be an arbitrary point in \( \tilde{\Omega} \). Then there exist a ball \( B(\underline{y}, R(\underline{y})) \) of radius \( R(\underline{y}) \) centered at \( \underline{y} \) and a positive constant \( M(\underline{y}) \), such that

\[
\left| \partial_\underline{x}^{n-j} A_s(\underline{x}) \partial_\underline{x}^j \right| \leq M(\underline{y}) \frac{n!}{R^n(\underline{y})}, \quad \underline{x} \in B(\underline{y}, R(\underline{y})), \quad j = 0, \ldots, n, \quad s = 0, 1.
\]
It follows that
\[ |A_n(x)| \leq M(y) \frac{n + R(y) - 1}{R^n(y)}, \quad x \in B(y, R(y)), \]
and therefore the series (3) converges normally in
\[ \tilde{\Omega} = \bigcup_{y \in \Omega} (-R(y), R(y)) \times B(y, R(y)). \]

Note that \( \tilde{\Omega} \) is a \( x_0 \)-normal open neighbourhood of \( \Omega \) in \( \mathbb{R}^{m+1} \), i.e. for each \( x \in \tilde{\Omega} \) the line segment \( \{x + t : t \in \mathbb{R}\} \cap \tilde{\Omega} \) is connected and contains one point in \( \tilde{\Omega} \).

We thus have proved the following.

**Theorem 1** The function \( \text{CK}[A_0, A_1] \) given by
\[
\text{CK}[A_0, A_1](x) = A_0(x) + x_0A_1(x) - \sum_{n=2}^{\infty} \frac{(-x_0)^n}{n!} \left( \sum_{j=0}^{n-2} \partial_{x}^{n-j-1}A_0(x)\partial_x^{j+1} + \sum_{j=0}^{n-1} \partial_{x}^{n-j-1}A_1(x)\partial_x^j \right)
\]
(5)
is inframonogenic in a \( x_0 \)-normal open neighbourhood of \( \Omega \) in \( \mathbb{R}^{m+1} \) and satisfies conditions (1)-(2).

It is worth noting that if in particular \( A_1(x) = -\partial_x A_0(x) \), then
\[
\text{CK}[A_0, -\partial_x A_0](x) = \sum_{n=0}^{\infty} \frac{(-x_0)^n}{n!} \partial_x^n A_0(x),
\]
which is nothing else but the left monogenic extension (or CK-extension) of \( A_0(x) \). Similarly, it is easy to see that \( \text{CK}[A_0, -A_0\partial_x](x) \) yields the right monogenic extension of \( A_0(x) \) (see [2, 5, 13, 16, 17]).

Let \( P(k) \) \((k \in \mathbb{N}_0 \text{ fixed})\) denote the set of all \( \mathbb{R}_{n,m} \)-valued homogeneous polynomials of degree \( k \) in \( \mathbb{R}^m \). Let us now take \( A_0(x) = P_k(x) \in P(k) \) and \( A_1(x) = P_{k-1}(x) \in P(k-1) \). Clearly,
\[
\text{CK}[P_k, P_{k-1}](x) = P_k(x) + x_0P_{k-1}(x) - \sum_{n=2}^{k} \frac{(-x_0)^n}{n!} \left( \sum_{j=0}^{n-2} \partial_{x}^{n-j-1}P_k(x)\partial_x^{j+1} + \sum_{j=0}^{n-1} \partial_{x}^{n-j-1}P_{k-1}(x)\partial_x^j \right),
\]
since the other terms in the series (5) vanish. Moreover, we can also claim that $\mathbf{CK}[P_k, P_{k-1}] (x)$ is a homogeneous inframonogenic polynomial of degree $k$ in $\mathbb{R}^{m+1}$.

Conversely, if $P_k(x)$ is a homogeneous inframonogenic polynomial of degree $k$ in $\mathbb{R}^{m+1}$, then $P_k(x) |_{x_0 = 0} \in \mathbb{P}(k)$, $\partial_{x_0} P_k(x) |_{x_0 = 0} \in \mathbb{P}(k-1)$ and obviously $\mathbf{CK}[P_k |_{x_0 = 0}, \partial_{x_0} P_k |_{x_0 = 0}] (x) = P_k(x)$.

Call $I(k)$ the set of all homogeneous inframonogenic polynomials of degree $k$ in $\mathbb{R}^{m+1}$. Then $\mathbf{CK}[\ldots]$ establishes a bijection between $\mathbb{P}(k) \times \mathbb{P}(k-1)$ and $I(k)$.

It is easy to check that

$$P_k(x) = P_k(\partial_{\bar{u}}) \frac{\langle x, u \rangle^k}{k!}, \quad P_k(x) \in \mathbb{P}(k),$$

where $P_k(\partial_{\bar{u}})$ is the differential operator obtained by replacing in $P_k(u)$ each variable $u_j$ by $\partial_{u_j}$. Therefore, in order to characterize $I(k)$, it suffices to calculate $\mathbf{CK}[\langle x, u \rangle^k e_A, 0]$ and $\mathbf{CK}[0, \langle x, u \rangle^{k-1} e_A]$ with $u \in \mathbb{R}^m$.

A simple computation shows that

$$\mathbf{CK}[\langle x, u \rangle^k e_A, 0](x) = \langle x, u \rangle^k e_A$$

$$- \sum_{n=2}^{k} \binom{k}{n} (-x_0)^n \langle x, u \rangle^{k-n} \left( \sum_{j=0}^{n-2} u_j^{n-j-1} e_A u_j^{j+1} \right),$$

$$\mathbf{CK}[0, \langle x, u \rangle^{k-1} e_A](x) = x_0 \langle x, u \rangle^{k-1} e_A$$

$$- \frac{1}{k} \sum_{n=2}^{k} \binom{k}{n} (-x_0)^n \langle x, u \rangle^{k-n} \left( \sum_{j=0}^{n-1} u_j^{n-j-1} e_A u_j^{j} \right).$$

Acknowledgments

The second author was supported by a Post-Doctoral Grant of Fundação para a Ciência e a Tecnologia, Portugal (grant number: SFRH/BPD/49200/2008).

References


