Precoded Modulations for Outage Probability Minimization on Block Fading channels

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Abstract

We study precoded modulations for the outage probability minimization of block fading channels. This paper establishes interesting upper bounds on the outage probability, whose simple expressions allow to determine which precoding matrix minimizes these upper bounds. Through this optimization, the gap between the outage probability corresponding with a discrete input alphabet and the best outage achieved by an i.i.d. Gaussian alphabet is almost closed.

1 Introduction

A block fading (BF) channel [1] is a useful model for all channels that consist of parallel sub-channels (e.g. via time-interleaving, frequency hopping, OFDMA or cooperative communications). The outage probability limit is a fundamental and achievable lower bound on the average word error rate of coded systems [1], [10]. By choosing a well designed precoding matrix P, the outage probability can be minimized [5]. Because no closed form expression for the outage probability is known, only a brute force optimization could be applied. A unitary $B \times B$ precoding matrix has B^2 degrees of freedom, so that the optimization of the outage probability is multivariate. In a brute force optimization, Monte Carlo simulations are required to take into account the distribution of all fading gains when evaluating the outage probability, which is often intractable.

In order to simplify the optimization of the outage probability, we establish upper bounds on the outage probability that can be optimized without performing Monte Carlo simulations. In [5], such upper bounds on the outage probability were established for BF channels with i.i.d. Gaussian input alphabets and discrete input alphabets *without* precoding. Also in [5], an illustration, without proof, of upper bounds on the outage probability of BF channels with $B = 2$ and discrete input alphabets *with* precoding was given. Here, we formally proof the latter case for high signal-to-noise ratio (SNR). The proof is assisted by a new channel model giving more insight.

2 A new channel equation for BF channels

The transmitter output is a real or complex vector $\mathbf{x} = [\mathbf{x}(1), \dots \mathbf{x}(B)]$ where $\mathbf{x}(b) = [x(b)_1, \dots, x(b)_{\frac{N}{B}}]$ is the b-th part of the transmitted vector. The received vector and the noise vector are similarly represented. The channel is memoryless with additive white Gaussian noise and multiplicative real fading (Rayleigh distributed). The fading coefficients are only known at the decoder side where the received signal vector is $\mathbf{y}(b) = \alpha_b \mathbf{x}(b) + \mathbf{w}(b), \quad b = 1, \dots, B$, where the fading coefficient α_b is independent and identically distributed (i.i.d.) from block to block. The noise vector $\mathbf{w}(b)$ consists of N/B independent noise samples which are complex Gaussian distributed, $w(b)_n \sim \mathcal{CN}(0, \frac{1}{\gamma})$, where γ is the average signal-to-noise ratio.

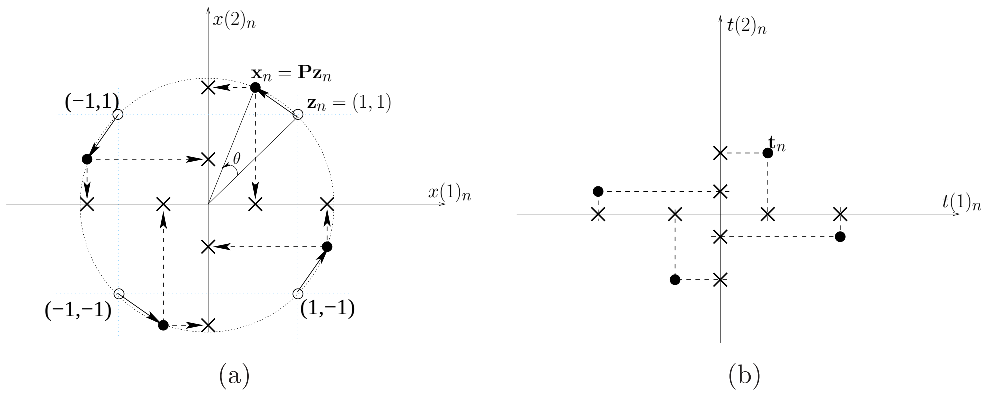


Figure 1: Displaying the rotation at the transmitter (a) ($\mathbf{x}_n$) and at the receiver (without noise) (b) ($\mathbf{t}_n$) for $B = 2$. The empty (filled) circles represent Ω_z (Ω_x). The components of $\mathbf{t}_n$ are obtained by scaling the components $\mathbf{x}_n$ by their respective fading gain (here $\alpha_2 < \alpha_1$). The crosses on the coordinate axes are the transmitted and received vector components, respectively.

It was shown in [2] that component interleaving combined with linear precoding can yield important gains in BF channels. In that case, each string of m bits is mapped to one of $M = 2^m$ points belonging to a B-dimensional real or complex space; the corresponding B-dimensional M-point constellation Ω_z is denoted M-$\mathcal{R}^B$ or M-$\mathcal{C}^B$, respectively. Denoting as $\mathbf{z}_n = [z(1)_n, \ldots, z(B)_n]^T$ the B-dimensional vector that results from mapping the n-th string of m coded bits, the linear precoding involves the computation

$$\mathbf{x}_n = P\mathbf{z}_n,\ n = 1, \ldots, \frac{N}{B}, \tag{1}$$

where the symbols $\{x(b)_n, b = 1, \ldots, B\}$ are the components of the multidimensional symbol $\mathbf{x}_n$ (we similarly consider $\mathbf{y}_n$ and $\mathbf{w}_n$), which belongs to the B-dimensional M-point constellation Ω_x. A precoding matrix P that is unitary is an obvious choice because it does not decrease the capacity of a Gaussian channel. In this paper, we will restrict our study to real precoding matrices, hence P is orthogonal. When $B = 2$, P is a rotation matrix with rotation angle θ.

Fig. 1 illustrates the effect of a rotation for $B = 2$ when a 4-$\mathcal{R}^2$ constellation is used as Ω_z. In that case, Ω_x is a rotated version of $\Omega_z =$ 4-$\mathcal{R}^2$. When received at the destination, the components of $\mathbf{x}_n$ are affected by their corresponding fading gain and noise. When only considering the fading, we consider $\mathbf{t}_n = \boldsymbol{\alpha} \cdot \mathbf{x}_n = [\alpha_1 x(1)_n, ..., \alpha_B x(B)_n]$, which is shown at the right side in Fig. 1. The symbol $\mathbf{t}_n = [t(1)_n, ..., t(B)_n]$ belongs to the constellation Ω_t, which we denote as the *faded* constellation. Note that this constellation is different for each codeword, because the fading point $\boldsymbol{\alpha}$ changes.

This system is better modelled by the channel equation

$$\mathbf{y}_n = \mathbf{t}_n + \mathbf{w}_n, \quad n = 1, \ldots, \frac{N}{B}. \tag{2}$$

Depending on the set of fading points $\boldsymbol{\alpha}$ that is considered, the constellation Ω_t at the input of this Gaussian vector channel will vary. This new channel equation gives more insight and is important in the new proofs proposed here.

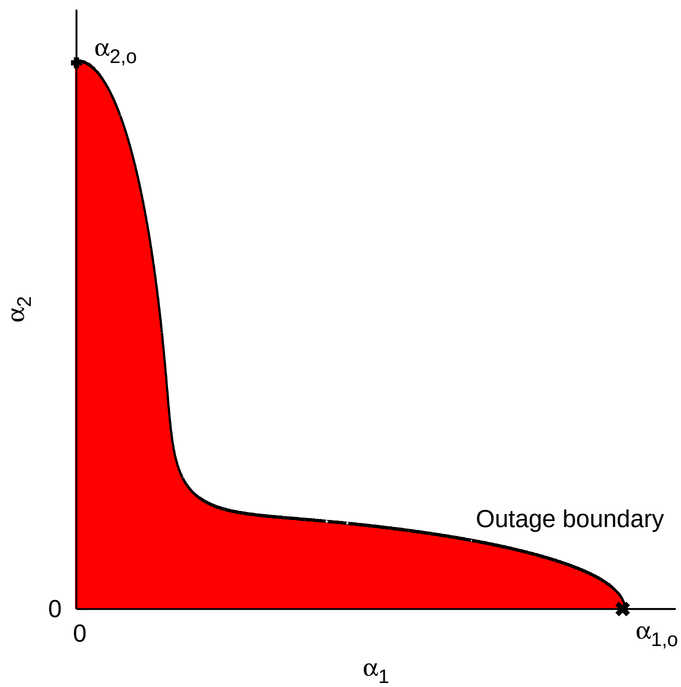


Figure 2: The outage boundary limits the region V_o (coloured red) in the fading space which corresponds to an information theoretic outage event. The points $\{\alpha_{b,o},\ b = 1, \ldots, B\}$ are shown for $B = 2$. The information rate is $R = 0.9$ bpcu. The average SNR is fixed to $\gamma = 8$dB.

3 Method to Upper Bound Outage Probability

For the remainder of the paper, we will drop the index n in the vectors $\mathbf{z}_n$, $\mathbf{x}_n$, $\mathbf{t}_n$, $\mathbf{y}_n$ and $\mathbf{w}_n$, as the time index is not important when considering mutual information. We write random variables using upper case letters corresponding to the lower case letters used for their realizations. The mutual information $I(\boldsymbol{\alpha}, \gamma, P)$ at a certain fading point $\boldsymbol{\alpha}$ between the transmitted B-dimensional symbol $\mathbf{x}$ and the corresponding received vector $\mathbf{y}$ is given by [7], [6]

$$I(\boldsymbol{\alpha}, \gamma, P) = \frac{1}{B} I(\mathbf{X}; \mathbf{Y}|\boldsymbol{\alpha}, \gamma) = \frac{1}{B} I(\mathbf{T}; \mathbf{Y}|\boldsymbol{\alpha}, \gamma), \tag{3}$$

where the last term corresponds with Eq. (2), and where the scaling factor $\frac{1}{B}$ is added because the B blocks in the channel timeshare a time-interval [4, Section 9.4].

The outage probability is the probability that the instantaneous mutual information is less than the transmitted rate. In other words, the outage probability corresponds to a set of fading gains where the channel is sufficiently bad (the channel is in *outage*):

$$P_{out}(\gamma, \Omega_z, P, R) = \int_{\boldsymbol{\alpha} \in V_o} p(\boldsymbol{\alpha}) d\boldsymbol{\alpha}, \tag{4}$$

where V_o is the region of fading gains in the fading space [3] such that $I(\boldsymbol{\alpha}, \gamma, P, \Omega_z) < R$. The region V_o is limited by an *outage boundary* $B_o(\gamma, P, \Omega_z, R)$, defined by $I(\boldsymbol{\alpha}, \gamma, P, \Omega_z) = R$ (Fig. 2). Here, we describe the precoder P and constellation Ω_z minimizing the outage probability $P_{out}(\gamma, P, \Omega_z, R)$.

Definition 1 *We define $\alpha_{b,o}$ by the magnitude of the intersection between the outage boundary and the axis α_b. More precisely, $I\left(\boldsymbol{\alpha}|_{\alpha_i=0,i\neq b,\alpha_b=\alpha_{b,o}},\gamma,P\right)=R$. By convention, $\alpha_{b,o}=+\infty$ if the axis α_b is an asymptote for the outage boundary.*

In the remainder of the paper, we denote the points $\boldsymbol{\alpha}|_{\alpha_i=0,i\neq b,\alpha_b=\alpha_{b,o}}$ by $\boldsymbol{\alpha}_{b,o}$. The defined points are illustrated in Fig. 2 for $B=2$. The points $\{\boldsymbol{\alpha}_{b,o}, b=1,\ldots,B\}$ correspond to the case that all fading gains are zero, except one, whose value is the scaling factor of the projection of the multidimensional constellation on the b-th coordinate axis $x(b)$, so that the mutual information between the B-dimensional symbol $\mathbf{x}$ and the corresponding received signal vector $\mathbf{y}$ is equal to the spectral efficiency BR (see Eq. (3)). In other words, if the projection of the multidimensional constellation on each coordinate axis yields the same set of points, then the magnitudes of the points $\{\boldsymbol{\alpha}_{b,o}, b=1,\ldots,B\}$ are equal. This is satisfied under mild symmetry conditions of Ω_x and we denote the projection of Ω_x on either coordinate axes by $\mathcal{S}_p$, where p stands for projection. The magnitudes $\{\alpha_{b,o}, b=1,\ldots,B\}$ are equal, so that we denote them by α_o.

Note that multidimensional constellations satisfying these symmetry conditions have an interesting property. For these constellations, the function $I\left(X(b);Y(b)|\alpha_b=\alpha,\gamma\right)$, which is the mutual information of a point-to-point channel with fading coefficient $\alpha_b=\alpha$, average SNR γ and with discrete input $X(b)$, does not depend on b. As a consequence, we will represent this mutual information by $I_{\mathcal{S}_p}\left(\alpha^2\gamma,P\right)$.

From Eq. (4), it is clear that the outage probability can be upper bounded by P_{up} corresponding to a boundary B_{up}, outer bounding $B_o(\gamma,P,\Omega_z,R)$:

$$P_{up}=\int_{\boldsymbol{\alpha}\in V_{up}} p(\boldsymbol{\alpha})d\boldsymbol{\alpha}, \tag{5}$$

where V_{up} is the region limited by B_{up}.

In the following section, we will determine a boundary with a simple shape outer bounding $B_o(\gamma,P,\Omega_z,R)$, which is then much easier to optimize. This can be done by determining a surface in the fading space, $U(\boldsymbol{\alpha})=0$, satisfying

$$I\left(\boldsymbol{\alpha},\gamma,P\right)\geq R, \text{ for all } \boldsymbol{\alpha} \text{ satisfying } U(\boldsymbol{\alpha})=0. \tag{6}$$

Here, the new channel equation, Eq. (2), can be used. By showing that for all $\boldsymbol{\alpha}$ satisfying $U(\boldsymbol{\alpha})=0$, the constellation Ω_t is distorted in such a way that $I(\mathbf{T};\mathbf{Y}|\boldsymbol{\alpha},\gamma)>BR$, it is proved that $U(\boldsymbol{\alpha})=0$ outer bounds $B_o(\gamma,P,\Omega_z,R)$.

For example, for $B=2$, we will prove that a circular arc touching $B_o(\gamma,P,\Omega_z,R)$ at $\alpha_1=\alpha_o$ (on the horizontal axis) and $\alpha_2=\alpha_o$ (on the vertical axis) satisfies Eq. (6).

4 Upper Bounds on the outage probability with linear precoding

In [5], it is proved that the outage boundary for a channel with an i.i.d. Gaussian input alphabet or a discrete input alphabet without precoding is outer bounded by the hypersurface of a B-hypersphere touching it at $\boldsymbol{\alpha}_{b,o}$, $\forall\, b$. Here, we prove that this is also true for a discrete input alphabet with linear precoding at high SNR.

Proposition 1 *On a BF channel at high SNR, with a discrete input alphabet and with linear precoding, the outage boundary $B_o(\gamma,P,\Omega_z,R)$ is outer bounded by the hypersurface of the B-hypersphere $\alpha_1^2+\alpha_2^2+\ldots+\alpha_B^2=\alpha_o^2$ touching it at the axes of the fading space.*

Proof 1 *Recall that* $I(\boldsymbol{\alpha}, \gamma, P) = \frac{1}{B} I(\mathbf{T}; \mathbf{Y}|\boldsymbol{\alpha}, \gamma)$ *(Eq. (3)). In [9] and [11], it is proven that for high SNR, maximizing (minimizing) the minimal Euclidean distance* d_{min} *of a constellation maximizes (minimizes) the mutual information on a Gaussian channel. For high SNR, the mutual information is [11]*

$$I(\mathbf{T}; \mathbf{Y}|\boldsymbol{\alpha}, \gamma) \approx m - 2\frac{K\pi}{2^m d_{min}^2(\boldsymbol{\alpha})} Q(d_{min}(\boldsymbol{\alpha})\sqrt{\gamma/2}), \tag{7}$$

where $d_{min}(\boldsymbol{\alpha})$ *is the minimal distance of the constellation* Ω_t *and* K *is the number of pairs of points at minimum distance in the constellation* Ω_t*.*

Consider two points $\mathbf{x}^{(i)} = (u_1^{(i)}, \ldots, u_B^{(i)})$ *and* $\mathbf{x}^{(j)} = (u_1^{(j)}, \ldots, u_B^{(j)})$ *from the constellation* Ω_x*. The corresponding points* $\mathbf{t}^{(i)}$ *and* $\mathbf{t}^{(j)}$ *from the faded constellation* Ω_t *have a squared Euclidean distance given by*

$$\left|\mathbf{t}^{(i)} - \mathbf{t}^{(j)}\right|^2 = \sum_{b=1}^{B} \alpha_b^2 \left(u_b^{(i)} - u_b^{(j)}\right)^2. \tag{8}$$

Let us denote by b^* *the value of* $b \in \{1, \ldots, B\}$ *for which* $\left(u_b^{(i)} - u_b^{(j)}\right)^2$ *is minimum. For the B-hypersphere* $\sum_{b=1}^{B} \alpha_b^2 = \alpha_o^2$*, we obtain*

$$\left|\mathbf{t}^{(i)} - \mathbf{t}^{(j)}\right|^2 \geq \alpha_o^2 \left(u_{b^*}^{(i)} - u_{b^*}^{(j)}\right)^2, \tag{9}$$

where equality in Eq. (9) is achieved when $\alpha_{b^*} = \alpha_o$*. When* b^* *is not unique, equality in Eq. (9) holds when* $\alpha_{b^*} = \alpha_o$ *holds for any* b^* *that minimizes* $\left(u_b^{(i)} - u_b^{(j)}\right)^2$*. Hence, the minimum distance* $d_{min}(\boldsymbol{\alpha})$ *for the constellation* Ω_t *is given by*

$$d_{min}(\boldsymbol{\alpha}) = \alpha_o \min_{\substack{i,j \in \{1,\ldots,M\} \\ i \neq j}} \min_{b \in \{1,\ldots,B\}} \left|u_b^{(i)} - u_b^{(j)}\right| \tag{10}$$

As the constellation Ω_x *is such that its projection on either coordinate axis yields the same set of points* $\mathcal{S}_p$*, the minimum distance* $d_{min}(\boldsymbol{\alpha})$ *is achieved for either fading gain equal to* α_o *(and the remaining* $B-1$ *fading gains equal to* 0*).*

The proof for low SNR is outside the scope of this paper, but we conjecture that the hypersurface of the considered hypersphere outer bounds the outage boundary for all SNRs. This outer boundary corresponds with an upper bound on the outage probability. Minimizing this upper bound is simply achieved by minimizing α_o.

For completeness, we note that the precoded constellations achieve full diversity for a rate R, provided that 2^{BR} does not exceed the number of points contained in the projection of Ω_x on a coordinate axis [5]. When considering the 4-$\mathcal{R}^2$ constellation from Fig. 1(a), $\mathcal{S}_p$ contains only 2 points when the rotation angle θ is a multiple of $\pi/2$, and 4 points otherwise. Therefore, the maximum rate that corresponds to full diversity is $R = 0.5$ when the rotation angle θ is a multiple of $\pi/2$, and $R = 1$ otherwise.

5 Minimizing the outage probability of precoded constellations

In the previous section, we proved for high SNR that the outage boundary of block fading channels with precoded constellations is outer bounded by a hypersurface of a region $V_{up}(\alpha_o)$, a B-hypersphere with center in the origin and touching the outage boundary at the axes of the fading space. This hypersurface corresponds to an upper bound $P_{up}(\alpha_o)$ on the outage probability of the channel (see Eq. (5)). Instead of minimizing the actual outage probability, it is easier to minimize the upper bound on the outage probability. This optimization allows the actual outage probability to closely approach a lower bound on the outage probability, i.e., the outage probability corresponding to an i.i.d. Gaussian input alphabet, which is illustrated in the numerical results.

The B-hypersphere is completely determined by one variable, its radius α_o. From Eq. (5), it is clear that the region $V_{up}(\alpha_o)$ has to be made as small as possible to minimize $P_{up}(\alpha_o)$. Therefore, the optimization target is to minimize the radius α_o.

Because $I_{\mathcal{S}_p}(\alpha_o^2\gamma, P) = BR$, the minimization of α_o is achieved by selecting the constellation $\mathcal{S}_p$ requiring the least energy to achieve a rate BR. This involves a proper selection of both the constellation Ω_z and the precoding matrix P. Note that this optimization is much simpler than the direct minimization of the outage probability, because it is hard to evaluate, especially when the number of fading gains and constellation points is large. Furthermore, no insight is gained by the latter approach, so that it would not be clear which constellation Ω_z should be taken.

In this section, we will show that the radius of the outer boundary, α_o, can be minimized by combining a simple optimization of the precoding matrix P with a constellation expansion.

5.1 Optimization of the precoding matrix

Let us denote by $\mathcal{O}$ the set of parameters from P over which we will minimize α_o. For $B = 2$, the only degree of freedom is the rotation angle. For larger B, more degrees of freedom can be exploited to minimize α_o. For the numerical results, we restrict ourselves to $B = 2$.

The mutual information of $\mathcal{S}_p$ can be rewritten as $I_{\mathcal{S}_p}(\alpha_o^2\gamma, \mathcal{O})$, which yields $\alpha_o^2 = \frac{I_{\mathcal{S}_p}^{-1}(BR,\mathcal{O})}{\gamma}$. Changing the value of $\mathcal{O}$ (e.g. the rotation angle θ for $B = 2$) will change the distances between the points in $\mathcal{S}_p$ and so change its mutual information. For a fixed spectral efficiency R and fixed average SNR γ, minimizing the radius yields the optimization criterion

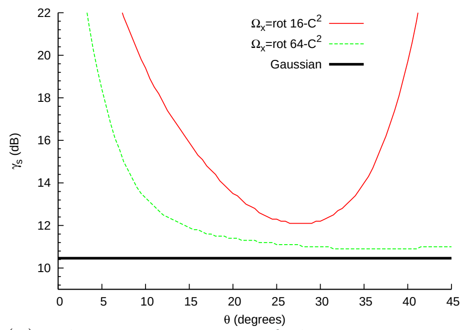
$$\mathcal{O}_{opt} = \arg\ \min_{\mathcal{O}} I_{\mathcal{S}_p}^{-1}(\mathcal{O}, BR).$$

The optimization is performed by means of a simulation, due to the lack of closed form expressions of the mutual information. Fortunately, the constellation is one-dimensional, so the computational effort is minimal.

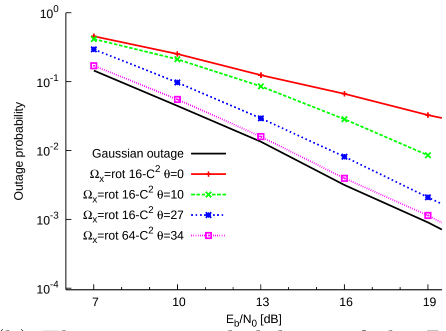
5.2 Optimization of the constellation Ω_z

As the number of information bits per channel use is $R = mR_c/B$, there are different combinations of m and R_c yielding the same R. The number of points in the constellation is $|\Omega_t|$. Increasing the constellation size of Ω_z will render a constellation Ω_t with more points. This higher order constellation may need less energy to achieve the same rate. However, the decoding complexity increases as well as the complexity of optimization, so that there is a trade-off between performance and complexity. The higher the constellation size, the smaller the horizontal SNR-gap between the outage probabilities corresponding to a precoded discrete input alphabet and i.i.d. Gaussian input alphabet. But the improvement in performance becomes smaller and smaller, as illustrated in Sec. 6.

Note that we limit the optimization of Ω_z to expanding Ω_z, so that we do not really optimize it. Fortunately, this expansion seems sufficient.



(a) The optimization of the rotation angle for complex symbols is shown.



(b) The outage probabilities of the BF channel with complex inputs are shown.

Figure 3: The optimization of θ and the optimized outage probabilities are shown when complex symbols are transmitted. The transmitted rate is $R = 1.8$ bpcu.

6 Numerical results

For the numerical results, we take $B = 2$ so that $\mathcal{O} = \theta$, and the optimization criterion for the upper bound on the outage probability is to find θ so that $I^{-1}_{\mathcal{S}_p}(\theta, BR)$ is minimized. Next, a constellation expansion is performed to further minimize the upper bound on the outage probability. In [5], the effect of precoding on real constellations was studied. Here, we focus on complex constellations, for which all the proofs in this paper are valid. This means that also for complex constellations, the outage boundary is outer bounded by a B-hypersphere, determined by one variable, its radius. We restrict our attention to real precoding matrices.

Assume that a transmission rate $R = 1.8$ bpcu is aimed. Initially, we take the constellation $\Omega_z = 16\text{-}\mathcal{C}^2$ ($R_c = 0.9$), which can be build as the Cartesian product of two 4-QAM constellations (16-$\mathcal{C}^2$=4-QAM$\times$4-QAM). First, we consider the optimization of the rotation angle θ. On the y-axis, we show the instantaneous SNR per symbol, $\gamma_s = \alpha_o^2\gamma$, so that $I_{\mathcal{S}_p}(\gamma_s, \theta) = BR$. The minimum SNR per symbol γ_s that is needed to transmit $R = 1.8$ bpcu for $\gamma = 8$dB is achieved by an i.i.d. complex Gaussian input alphabet:

$$\gamma_s = 2^{2R} - 1.$$

This fundamental minimum can be approached when using a precoded discrete input $\Omega_z = 16\text{-}\mathcal{C}^2$ with rotation angle $\theta = 27$ degrees. Now, we apply a constellation expansion to further reduce γ_s (see Fig. 3(a)). For example, γ_s for the rotated constellation $\Omega_z = 64\text{-}\mathcal{C}^2$ (8-QAM$\times$8-QAM, $R_c = 0.6$) approaches the theoretical minimum very closely for rotation angles within $[35, 45]$ degrees. The optimization of θ and the optimized outage probabilities are shown in Fig. 3. The optimized outage probability corresponding with a discrete input alphabet approaches the outage probability corresponding with an i.i.d. complex Gaussian input alphabet up to a few tenths of a dB.

7 Conclusions

We have studied the effect of linear precoding on the outage probability of block fading channels. We have analyzed the outage boundaries in the fading space and established outer boundaries with simple shapes which yield an easy optimization of the outage probability for a discrete constellation, for an arbitrary number of blocks in the fading channel. The combination of a constellation expansion and an optimized precoding matrix, has shown to be sufficient to closely approach the outage probability corresponding to an i.i.d. Gaussian input alphabet.

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