On the fundamental solution and integral formulae of a higher spin operator in several vector variables

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Abstract. The fundamental solution is constructed of a particular higher spin operator in $\mathbb{R}^m$, acting on functions taking values in an irreducible representation space for $\mathrm{Spin}(m)$ with highest weight $(k+\frac{1}{2}, l+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, with $k, l \in \mathbb{N}$ and $k \geq l$. Some basic integral formulae are stated.

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INTRODUCTION AND DEFINITIONS

Let $(e_1, \ldots, e_m)$ be an orthonormal basis for the Euclidean space $\mathbb{R}^m$ and denote by $\mathbb{C}_m$ the complex 2^m-dimensional Clifford algebra over $\mathbb{R}^m$, generated by the relations $e_i e_j + e_j e_i = -2\delta_{i,j}$, $i, j = 1, \ldots, m$. By identifying $(x_1, \ldots, x_m)$ with the real Clifford vector $x = \sum_{j=1}^m e_j x_j$, the space $\mathbb{R}^m$ is embedded in $\mathbb{C}_m$. The multiplication of two vectors x and y is given by $xy = -\langle x, y\rangle + x \wedge y$, with

$$\langle x, y\rangle = \sum_{j=1}^m x_j y_j \quad \text{and} \quad x \wedge y = \sum_{1\leq i<j\leq m} e_i e_j (x_i y_j - x_j y_i)$$

being the scalar-valued Euclidean inner product and the bivector-valued wedge product respectively.

The Pin group $\mathrm{Pin}(m)$ is the group consisting of products of unit vectors in $\mathbb{R}^m$. The Spin group $\mathrm{Spin}(m)$ is the subgroup of $\mathrm{Pin}(m)$ consisting of products of an even number of unit vectors in $\mathbb{R}^m$. For convenience we work in odd dimension $m = 2n+1$, in which case there is a unique spinor space $\mathbb{S}$. This space is an irreducible $\mathrm{Spin}(m)$-module and can be thought of as a minimal left ideal in $\mathbb{C}_m$.

The Dirac operator on $\mathbb{R}^m$ is given by $\partial_x = \sum_{j=1}^m e_j \partial_{x_j}$. It is an elliptic $\mathrm{Spin}(m)$-invariant first-order differential operator acting on $\mathbb{S}$-valued functions $f(x)$ on $\mathbb{R}^m$. A spinor-valued function f is monogenic in an open region $\Omega \subset \mathbb{R}^m$ if and only if it satisfies $\partial_x f = 0$ in Ω. For a detailed account of the theory of monogenic functions, so-called Euclidean Clifford analysis, we refer to e.g. [2, 10, 14]. More abstractly, on manifolds, the Dirac operator can be regarded as the simplest case of the unique (up to a multiple) conformally invariant elliptic first-order operator acting between appropriate sections of spin bundles, see e.g. [6, 11, 17]. In Clifford analysis these operators are studied from a function theoretical point of view, considering functions on $\mathbb{R}^m$ instead of sections. Another example of such operators are the generalized Rarita-Schwinger operators $\mathscr{R}_k$, acting on functions taking values in irreducible $\mathrm{Spin}(m)$-representation spaces with highest weight $(k+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ with $k \in \mathbb{N}$, see e.g. [7, 8]. This notion has been generalized in [3, 4], where the operator $\mathscr{Q}_{k,l}$ is studied, acting on functions with values in the irreducible $\mathrm{Spin}(m)$-module with highest weight $(k+\frac{1}{2}, l+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ with $k, l \in \mathbb{N}$ and $k \geq l$.

Both representations can be described by vector spaces of polynomials within the framework of Clifford analysis, see e.g. [9]. For example, the space $\mathscr{H}_k$ of $\mathbb{C}$-valued harmonic homogeneous polynomials of degree $k \in \mathbb{N}$ corresponds to the irreducible $\mathrm{Spin}(m)$-module with highest weight $(k, 0, \ldots, 0)$, and the space $\mathscr{M}_k$ of $\mathbb{S}$-valued monogenic homogeneous polynomials of degree k forms an irreducible representation of $\mathrm{Spin}(m)$ with highest weight $(k+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Both notions are generalized in the following definitions, where $N \in \mathbb{N}$ and ∂_i is short for the Dirac operator ∂_{u_i}.

Definition 1. *A function $f : \mathbb{R}^{Nm} \to \mathbb{C}$, $(u_1, \ldots, u_N) \mapsto f(u_1, \ldots, u_N)$ is simplicial harmonic if the following conditions are satisfied: $\langle \partial_i, \partial_j\rangle f = 0$ $(i, j = 1, \ldots, N)$ and $\langle u_i, \partial_j\rangle f = 0$ $(1 \leq i < j \leq N)$. The vector space of $\mathbb{C}$-valued simplicial harmonic polynomials, λ_i-homogeneous in the variable u_i, will be denoted by $\mathscr{H}_{\lambda_1,\ldots,\lambda_N}$ (with $\lambda_1 \geq \ldots \geq \lambda_N \geq 0$).*

Definition 2. *A function* $f : \mathbb{R}^{Nm} \to \mathbb{S}$, $(u_1,\ldots,u_N) \mapsto f(u_1,\ldots,u_N)$ *is simplicial monogenic if the following conditions are satisfied:* $\partial_i f = 0$ $(i = 1,\ldots,N)$ *and* $\langle u_i, \partial_j \rangle f = 0$ $(1 \le i < j \le N)$*. The vector space of* $\mathbb{S}$*-valued simplicial monogenic polynomials,* λ_i*-homogeneous in the variable* u_i*, will be denoted by* $\mathscr{S}_{\lambda_1,\ldots,\lambda_N}$ *(with* $\lambda_1 \ge \ldots \ge \lambda_N \ge 0$*).*

Take $N = 2$ and $(u_1,u_2) = (u,v)$ in both definitions. In [3], the operator $\mathscr{Q}_{k,l}$ mentioned before was constructed by means of the following decomposition:

Proposition 1. *For any pair of integers* $k \ge l \ge 0$ *with* $k > 0$*, one has*

$$\mathscr{H}_{k,l} \otimes \mathbb{S} \;=\; \mathscr{S}_{k,l} \oplus (1-\delta_{l,0})\nu_{k,l}\,\mathscr{S}_{k,l-1} \oplus (1-\delta_{k,l})\mu_{k,l}\,\mathscr{S}_{k-1,l} \oplus (1-\delta_{l,0})\kappa_{k,l}\,\mathscr{S}_{k-1,l-1},$$

with $\nu_{k,l}$*,* $\mu_{k,l}$ *and* $\kappa_{k,l}$ *the respective embedding maps of the spaces* $\mathscr{S}_{k,l-1}$*,* $\mathscr{S}_{k-1,l}$ *and* $\mathscr{S}_{k-1,l-1}$ *in* $\mathscr{H}_{k,l} \otimes \mathbb{S}$*.*

If we denote by $\pi_{k,l}$ the projection operator $\pi_{k,l} : \mathscr{H}_{k,l} \otimes \mathbb{S} \to \mathscr{S}_{k,l}$, we have

Definition 3. *For all integers* $k \ge l \ge 0$ *with* $k > 0$*, there are (up to a multiplicative constant) unique invariant first-order differential operators* $\mathscr{Q}_{k,l}$ *defined by*

$$\mathscr{Q}_{k,l} : \mathscr{C}^\infty(\mathbb{R}^m, \mathscr{S}_{k,l}) \;\to\; \mathscr{C}^\infty(\mathbb{R}^m, \mathscr{S}_{k,l}),\; f(x;u,v) \;\mapsto\; \pi_{k,l}(\partial_x f)(x;u,v).$$

Explicitly,

$$\mathscr{Q}_{k,l} f = \left(\mathbf{1} + \frac{u\,\partial_u}{m+2k-2}\right)\left(\mathbf{1} + \frac{v\,\partial_v}{m+2l-4}\right)\partial_x f.$$

Remark 1. *In case* $l = 0$*, the operator* $\mathscr{Q}_{k,0}$ *reduces to the Rarita-Schwinger operators* $\mathscr{R}_k$*, as defined in [7].*

This contribution fits into the encompassing framework in which we are carrying out a thorough study of the higher spin Dirac operators $\mathscr{Q}_{k,l}$. It aims at constructing their fundamental solution which allows for establishing basic integral formulae such as Stokes' theorem, the Cauchy-Pompeiu representation theorem and the Cauchy integral formula for their null solutions, which are also stated. In their turn these integral formulae will prove to be essential to developing a function theory associated to these higher spin operators.

CONFORMAL INVARIANCE IN CLIFFORD ANALYSIS

The higher spin operators $\mathscr{Q}_{k,l}$ are not just Spin(m)-invariant, they are also conformally invariant, see e.g. [11, 16]. The conformal group can be described by the Vahlen group $V(m)$ of 2×2 matrices with entries in $\mathbb{C}_m$, defined as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(m) \;\Leftrightarrow\; \begin{cases} a,b,c,d \in \Gamma(m) \cup \{0\}, \\ ab^*, cd^*, d^*b, c^*a \in \mathbb{R}^m, \\ ad^* - bc^* = \pm 1, \end{cases}$$

with $\Gamma(m)$ the Clifford group. See e.g. [1, 15] for a detailed account of conformal invariance in Clifford analysis. In order to construct the fundamental solution of $\mathscr{Q}_{k,l}$, we use a general theorem of [16] about conformally invariant differential operators, adapted to our case of operators acting on $\mathscr{C}^\infty(\mathbb{R}^m, \mathscr{S}_{k,l})$. Let L be the standard representation of Pin(m) acting on the values $\mathscr{S}_{k,l}$.

Theorem 1. *If* f *is a null solution of* $\mathscr{Q}_{k,l}$*, so is the transformed function* $(g \cdot f)$*, defined by*

$$(g \cdot f)(x) = |cx+d|^{-m+1} L\left(\frac{(cx+d)^*}{|cx+d|}\right) f((ax+b)(cx+d)^{-1}; u, v)$$

with $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *an element of the conformal group* $V(m)$*.*

CONSTRUCTION OF THE FUNDAMENTAL SOLUTION

In this section it will become clear that the construction of the fundamental solution of the higher spin operators $\mathscr{Q}_{k,l}$ needs results from a variety of disciplines such as Clifford analyis, distribution theory and group representation theory, and moreover involves a lot of technicalities. We present a concise but rather complete view on this construction. Denote by A_m the surface area of the unit sphere in $\mathbb{R}^m$. Recall that the Cauchy kernel $E(x) = -(1/A_m)(\bar{x}/|x|^m)$ satisfies $\partial_x E(x) = \delta(x)$ in distributional sense. Choose an arbitrary polynomial P in $\mathscr{S}_{k,l}$ and let $C_{k,l}$ be a constant, which will be specified later. Inspired by Theorem 1, we define:

$$E_{k,l}(x;u,v) \ = \ C_{k,l}|x|^{-m+1}L(x/|x|)P(u,v) \ = \ C_{k,l}x|x|^{-(m+2k+2l)}P(xux,xvx) \tag{1}$$

We will prove that $E_{k,l}(x;u,v)$ satisfies $\mathscr{Q}_{k,l}E_{k,l}(x;u,v) = \delta(x)P(u,v)$ in distributional sense. Instead of taking an arbitrary $P \in \mathscr{S}_{k,l}$, it is convenient to work with the following generator of $\mathscr{S}_{k,l}$, see [9] for more information:

$$P_{k,l}(u,v) := \langle u,\tau_1\rangle^{k-l}\left[\det\begin{pmatrix} \langle u,\tau_1\rangle & \langle u,\tau_2\rangle \\ \langle v,\tau_1\rangle & \langle v,\tau_2\rangle \end{pmatrix}\right]^l I_{12} \ \in \ \mathscr{S}_{k,l} \tag{2}$$

with τ_1 and τ_2 null vectors in $\mathbb{C}^m$, i.e. $\tau_1^2 = \tau_2^2 = 0$, the idempotent $I_{12} = \tau_1\bar{\tau}_1\tau_2\bar{\tau}_2$. After calculations involving Definition 3 and some well-known identities in Clifford analysis, we find for $\alpha \in \mathbb{C}$:

$$\mathscr{Q}_{k,l}(|x|^{\alpha-2(k+l)}xP_{k,l}(xux,xvx)) = -(\alpha+m)|x|^{\alpha-2(k+l)}\pi_{k,l}(P_{k,l}(xux,xvx)). \tag{3}$$

If $\alpha = -m$, then $|x|^{-m-2(k+l)}xP_{k,l}(xux,xvx)$ belongs to the kernel of $\mathscr{Q}_{k,l}$ and shows a pointwise singularity of homogeneity degree $-m+1$ at the origin. In order to take the limit $\alpha \to -m$, it is necessary to recall some basic facts about distributions related to Riesz potentials in $\mathbb{R}^m$, see e.g. [12, 13]. We show in [5] that the relation (3) remains valid in distributional sense if $\Re(\alpha) > -m-1$. Furthermore,

$$\mathscr{Q}_{k,l}(|x|^{-m-2(k+l)}xP_{k,l}(xux,xvx)) = -\frac{\pi^{\frac{m}{2}}2^{-2(k+l)+1}}{\Gamma(\frac{m}{2}+k+l)(k+l)!}\pi_{k,l}(\Delta_x^{k+l}P_{k,l}(xux,xvx))\delta(x). \tag{4}$$

Denote by $\mathscr{P}_h$ the space of h-homogeneous polynomials. To calculate the right-hand side of (4), define the operator $\mathscr{R}_x : \mathscr{S}_{k,l} \to \mathscr{P}_{2(k+l)} \otimes \mathscr{H}_{k,l}$, $P_{k,l}(u,v) \mapsto P_{k,l}(xux,xvx)$. It can be proven that $\Delta_x^{k+l}\mathscr{R}_x$ is a Spin(m)-invariant map acting on $\mathscr{S}_{k,l}$. Hence it follows from Schur's lemma that, for some constant C, holds $\Delta_x^{k+l}P_{k,l}(xux,xvx) = CP_{k,l}(u,v)$, which is valid for every $P_{k,l}(u,v)$ in $\mathscr{S}_{k,l}$ since $\mathscr{S}_{k,l}$ is irreducible with respect to Spin(m). To dermine this constant C, we complexify u, v, τ_1 and τ_2. If $u = e_1 - ie_2$, $v = e_3 - ie_4$, $\tau_1 = e_1 + ie_2$ and $\tau_2 = e_3 + ie_4$, then we can show that

$$C \ = \ 2^{2(k+l)}(k+l)!\frac{\Gamma(\frac{m}{2}+k+l)}{\Gamma(\frac{m}{2}+k)}\frac{\Gamma(\frac{m-2}{2}+k)}{\Gamma(\frac{m-2}{2}+l)}\frac{\Gamma(\frac{m-4}{2}+l)}{\Gamma(\frac{m-4}{2})}.$$

The right-hand side of (4) can thus be simplified to

$$\mathscr{Q}_{k,l}(|x|^{-m-2(k+l)}xP_{k,l}(xux,xvx)) \ = \ -A_m\frac{(m-2)(m-4)}{(m+2k-2)(m+2l-4)}P_{k,l}(u,v)\delta(x) \ =: \ (C_{k,l})^{-1}P_{k,l}(u,v)\delta(x).$$

Hence,

$$E_{k,l}(x;u,v) \ := \ C_{k,l}|x|^{-m-2(k+l)}xP_{k,l}(xux,xvx) \ \in \ \mathscr{C}^\infty(\mathbb{R}^m\setminus\{0\},\mathscr{S}_{k,l})$$

satisfies

$$\mathscr{Q}_{k,l}E_{k,l}(x;u,v) = \delta(x)P_{k,l}(u,v). \tag{5}$$

If we define

$$e_{k,l}(x) \ := \ C_{k,l}|x|^{-m-2(k+l)}L(x/|x|) \ \in \ \mathscr{C}^\infty(\mathbb{R}^m\setminus\{0\},\mathrm{End}(\mathscr{S}_{k,l})),$$

we have $E_{k,l}(x;u,v) = e_{k,l}(x)P_{k,l}(u,v)$. It follows from (5) that $\mathscr{Q}_{k,l}e_{k,l}(x)P_{k,l}(u,v) = \delta(x)P_{k,l}(u,v)$. This relation depends on $P_{k,l} \in \mathscr{S}_{k,l}$. For this reason, we introduce the reproducing kernel $K_{k,l}$ satisfying

$$(K_{k,l}(u,v;u',v'),P_{k,l}(u,v))_{(u,v)} = P_{k,l}(u',v'),$$

with $(\cdot,\cdot)_{(u,v)}$ the Fischer inner product on $\mathscr{S}_{k,l}$ defined by $(P,Q)_{(u,v)} = \left[\bar{P}(\frac{\partial}{\partial u},\frac{\partial}{\partial v})Q\right]_0$. Now, if we put

$$E_{k,l}(x;u,v;u',v') \ := \ e_{k,l}(x)K_{k,l}(u,v;u',v') \ \in \ \mathscr{C}^\infty(\mathbb{R}^m\setminus\{0\},\mathrm{End}(\mathscr{S}_{k,l})),$$

we finally find

$$\mathscr{Q}_{k,l}E_{k,l}(x;u,v;u',v') = \delta(x)K_{k,l}(u,v;u',v'). \tag{6}$$

BASIC INTEGRAL FORMULAE

The basic integral formulae related to the operators $\mathcal{Q}_{k,l}$ can be deduced from Stokes' theorem for the Dirac operator, see e.g. [10]. Put $dx = dx_1 \wedge \ldots \wedge dx_m$ and $d\sigma_x = \sum_{j=1}^{m} (-1)^{j-1} e_j d\hat{x}_j$ with $d\hat{x}_j = dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_m$. In [5] we proved:

Theorem 2. *Let $\Omega' \subset \mathbb{R}^m$ and $\bar{\Omega} \subset \Omega'$. Then for $f, g \in \mathscr{C}^1(\Omega', \mathscr{S}_{k,l})$ one has:*

(i) Stokes' Theorem.

$$\int_{\Omega} \Big[-(\mathcal{Q}_{k,l} g(x), f(x))_{(u,v)} + (g(x), \mathcal{Q}_{k,l} f(x))_{(u,v)} \Big] dx \; = \int_{\partial\Omega} (g(x), \pi_{k,l}(d\sigma_x) f(x))_{(u,v)}.$$

(ii) Cauchy-Pompeiu Theorem. Let $y \in \Omega$. Then

$$f(y) + \int_{\Omega} e_{k,l}(x-y) \mathcal{Q}_{k,l} f(x) dx = \int_{\partial\Omega} e_{k,l}(x-y) \pi_{k,l}(d\sigma_x) f(x).$$

(iii) Cauchy integral formula. If $\mathcal{Q}_{k,l} f = 0$, then for $y \in \Omega$ one has:

$$f(y) = \int_{\partial\Omega} e_{k,l}(x-y) \pi_{k,l}(d\sigma_x) f(x).$$

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