Partial covers of $\mathrm{PG}(n, q)$

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Abstract

In this paper, we show that a set of $q+a$ hyperplanes, $q > 13$, $a \leq (q-10)/4$, that does not cover $\mathrm{PG}(n,q)$, does not cover at least $q^{n-1} - aq^{n-2}$ points, and show that this lower bound is sharp. If the number of non-covered points is at most q^{n-1}, then we show that all non-covered points are contained in one hyperplane. Finally, using a recent result of Blokhuis, Brouwer, and Szőnyi [3], we remark that the bound on a for which these results are valid can be improved to $a < (q-2)/3$ and that this upper bound on a is sharp.

1   Introduction

Let $\mathrm{PG}(n,q)$ denote the n-dimensional projective space over the finite field $\mathbb{F}_q$ with q elements, where $q = p^h$, p prime, $h \geq 1$. We denote the number of points in $\mathrm{PG}(n,q)$ by θ_n, i.e., $\theta_n = \frac{q^{n+1}-1}{q-1}$.

Let $\mathcal{C}$ be a family of $q + a$ hyperplanes of $\mathrm{PG}(n,q)$. Denote by $\mathcal{C}(P)$ the set of hyperplanes of $\mathcal{C}$ containing P. A $(q+a)$-cover $\mathcal{C}$ of $\mathrm{PG}(n,q)$ is a family $\mathcal{C}$ of $q + a$ different hyperplanes in $\mathrm{PG}(n,q)$ such that $|\mathcal{C}(P)| \geq 1, \forall P \in \mathrm{PG}(n,q)$. A *partial* $(q+a)$-*cover* $\mathcal{S}$ is a set of $q + a$ hyperplanes such that there is at least one point Q in $\mathrm{PG}(n,q)$ such that $|\mathcal{S}(Q)| = 0$. A point H for which $|\mathcal{S}(H)| = 0$, is called a *hole of* $\mathcal{S}$. We denote the set of holes of $\mathcal{S}$ by $\mathcal{H}_\mathcal{S}$.

A *blocking set* of $\mathrm{PG}(n,q)$ is a set B of points such that each hyperplane of $\mathrm{PG}(n,q)$ contains at least one point of B. A blocking set B is called *trivial* if it contains a line of $\mathrm{PG}(n,q)$. If a hyperplane contains exactly one point of a blocking set B in $\mathrm{PG}(n,q)$, it is called a *tangent hyperplane* to B, and a point P of B is called *essential* when it belongs to a tangent hyperplane to B. A blocking set B is called *minimal* when no proper subset of B is also a blocking set, i.e., when each point of B is essential.

It is clear that a cover of $\mathrm{PG}(n,q)$ is a dual blocking set. Dualizing the above definitions yields that a cover $\mathcal{C}$ is called *trivial* if it contains all hyperplanes through a certain $(n-2)$-space and *minimal* if no proper subset of $\mathcal{C}$ is a cover. A hyperplane π is *essential* to a cover $\mathcal{C}$ if there is a point $P \in \pi$ such that $\mathcal{C}(P) = \{\pi\}$.

The following reducibility results will be used throughout this article.

Result 1. *[7, Remark 3.3] A blocking set of size at most $2q$ in $\mathrm{PG}(2,q)$ is uniquely reducible to a minimal blocking set.*

Result 2. *[6, Corollary 1] A blocking set of size smaller than $2q$ in $\mathrm{PG}(n,q)$ is uniquely reducible to a minimal blocking set.*

In Theorem 7, we extend the following result of Blokhuis and Brouwer to general dimension.

Result 3. *[2] Let B be a blocking set in $\mathrm{PG}(2,q)$. If $|B| = 2q - s$, then there are at least $s + 1$ tangent lines through each essential point of B.*

Finally, for q a prime, we use the following result, proven by Blokhuis [1] for $n = 2$.

Result 4. *[5] Let B be a non-trivial blocking set in $\mathrm{PG}(n,p)$, where p is an odd prime. Then*
$$|B| \geq 3(p+1)/2.$$

2   Partial covers of $\mathrm{PG}(2,q)$

Throughout this section, $\mathcal{S}$ will denote a partial $(q + a)$-cover of $\mathrm{PG}(2,q)$, with $0 \leq a \leq (q - 10)/4$, $q > 13$.

Theorem 5. *If $|\mathcal{H}_\mathcal{S}| \leq q + a$, then $|\mathcal{H}_\mathcal{S}| \leq q$, and the holes are collinear.*

Proof. Let $|\mathcal{H}_\mathcal{S}| = x$. Suppose that there are three non-collinear points in $\mathcal{H}_\mathcal{S}$, otherwise the theorem is proven. The set $\mathcal{H}_\mathcal{S}$ can be covered by at most $(x+1)/2$ lines, denote the set of these lines by $\mathcal{L}$. Let $\mathcal{L}'$ be the minimal cover of $\mathcal{H}_\mathcal{S}$ contained in $\mathcal{L}$. The set $\mathcal{S} \cup \mathcal{L}'$ is a cover in $\mathrm{PG}(2,q)$. Since the size of $\mathcal{S} \cup \mathcal{L}'$ is at most $q + a + (q + a + 1)/2 \leq 2q$, there is a unique minimal cover $\mathcal{C}$ contained in $\mathcal{S} \cup \mathcal{L}'$ (Result 1).

Let $\ell_y \in \mathcal{L}'$ be a y-secant to $\mathcal{H}_\mathcal{S}$ with $y \leq (q - 3a - 1)/2$. Interchanging ℓ_y by y other lines gives, together with the lines of $\mathcal{S}$, another cover $\mathcal{C}'$, with $|\mathcal{C} \cup \mathcal{C}'| \leq q + a + (q + a + 1)/2 + (q - 3a - 1)/2 \leq 2q$. Hence, by the unique reducibility property, there is a unique minimal cover contained in $\mathcal{C} \cup \mathcal{C}'$, hence in $\mathcal{C} \cap \mathcal{C}'$. This minimal cover does not contain ℓ_y, hence $\mathcal{L}'$ contains only lines with at least $(q - 3a - 1)/2$ holes.

If $|\mathcal{L}'| = 1$, then the theorem is proven. Remark that if there is only one long secant and there are q holes, then $a = 0$.

Suppose that $|\mathcal{L}'| = z$. These z secants, together with the $q + a$ lines of $\mathcal{S}$, form a cover $\mathcal{C}''$. Then there is a line L in $\mathcal{L}'$ with less than $(q + a + 1 + \binom{z}{2})/z$ holes. Suppose to the contrary that any line in $\mathcal{L}'$ contains at least $(q + a + 1 + \binom{z}{2})/z$ holes, then there are at least $z(q + a + 1 + \binom{z}{2})/z - \binom{z}{2} = q + a + 1 > q + a$ holes, a contradiction. We construct a new cover by replacing this line L with less than $(q + a + 1 + \binom{z}{2})/z$ lines, one through each hole on L. In total, with the z secants and the lines of $\mathcal{S}$, this set of lines constitutes a cover $\mathcal{C}'''$ of size at most $q + a + z + (q + a + 1 + \binom{z}{2})/z$. If

$$q + a + z + (q + a + 1 + \binom{z}{2})/z \leq 2q, \tag{1}$$

the unique reducibility property (Result 1) shows that there is a minimal cover contained in $\mathcal{C}'' \cap \mathcal{C}'''$, which does not contain the line L. This implies that the line L was not essential to the cover $\mathcal{C}''$, a contradiction. It is easy to check that for $z \geq 2$ and $z < 9$, inequality (1) holds for $a \leq (q-10)/4$ and $q > 13$. Hence, there are at least 9 long secants essential to the minimal cover $\mathcal{C}''$. On each of these secants, there are at least $(q-3a-1)/2$ holes, hence we have at least $9(q-3a-1)/2 - 9 \cdot 8/2$ holes. But

$$9(q-3a-1)/2 - 36 > q + a$$

if $a < (7q-63)/25$. Since $a \leq (q-10)/4$, and $(q-10)/4 < (7q-63)/25$, the theorem follows. $\qquad\square$

Corollary 6. *Let q be a prime. If $|\mathcal{H}_\mathcal{S}| \leq q + a$, then $\mathcal{S}$ consists of q lines through the same point R and a lines $l_1, \ldots, l_a$, not through R.*

Proof. It follows from Theorem 5 that the holes are contained in one line, say M. Then the lines of $\mathcal{S}$, together with M, constitute a cover $\mathcal{C}$ of size $q+a+1 < 3(q+1)/2$. Result 4, together with Result 1, shows that the unique minimal cover contained in $\mathcal{C}$ is the set of all lines through a point R. It is clear that the line M is one of the lines through R. The other a lines are random, but do not contain R. $\qquad\square$

3   Partial covers of $\mathrm{PG}(n, q)$

Before extending the results of Section 2 to general dimension, we need the extension of Result 3.

Theorem 7. *The number of tangent hyperplanes through an essential point of a blocking set B of size $q+a+1$, $|B| \leq 2q$, in $\mathrm{PG}(n, q)$ is at least $q^{n-1} - aq^{n-2}$.*

Proof. The arguments of this proof are based on the proof of Proposition 2.5 in [8].

For $n = 2$, Result 3 proves this theorem. Assume by induction that the theorem holds for all dimensions $i \leq n-1$. Let B be a blocking set in $\pi = \mathrm{PG}(n, q)$. Since $|B| \leq 2q$, there is an $(n-2)$-space L in π that is skew to B. Let H be a hyperplane through L. Embed π in $\mathrm{PG}(2n-2, q)$. Let P be a $\mathrm{PG}(n-3, q)$, skew to π, in $\mathrm{PG}(2n-2, q)$. Then $\langle B, P \rangle$, the cone with vertex P and base B, is a blocking set with respect to the $(n-1)$-spaces of $\mathrm{PG}(2n-2, q)$. Let $H^* \neq H$ be a hyperplane through L only sharing one point Q with B. Since $|B|$ is at most $2q$, there are at least 2 tangent hyperplanes through L, hence H^* can be chosen different from H.

Let $\mathcal{S}$ be a regular $(n-2)$-spread through L and $\langle Q, P \rangle$ in W, the $(2n-3)$-dimensional space spanned by L and $\langle Q, P \rangle$. Using the André-Bruck-Bose construction (see [4]), this yields a projective plane $\mathrm{PG}(2, q^{n-1}) = \Pi^W$. The arguments of [8, Proposition 2.5] show that H defines a line ℓ in Π^W, only having essential points of the blocking set $\bar{B}$ of size $1 + (q+a)q^{n-2} = q^{n-1} + aq^{n-2} + 1$, where $\bar{B}$ is the blocking set in $\mathrm{PG}(2, q^{n-1})$, corresponding to $\langle B, P \rangle$. This number of points comes from $\langle Q, P \rangle$ at infinity, which is one point of the blocking set, and the $q + a$ affine points R_i of B, all on a cone $\langle R_i, P \rangle$ with q^{n-2} affine points. Result 3 shows that any essential point of $\bar{B}$ lies on at least $q^{n-1} - aq^{n-2}$

tangent lines to the blocking set $\bar{B}$ in Π^W. We will show that the number of tangent lines through an essential point of the blocking set $\bar{B}$ in Π^W is a lower bound on the number of tangent hyperplanes through an essential point of B in $\mathrm{PG}(n, q)$.

A tangent line through an affine essential point R of $\bar{B}$ corresponds to an $(n-1)$-space $\langle R, \Omega \rangle$, with Ω a spread element of $\mathcal{S}$. The space $\langle R, \Omega \rangle$ is not necessarily a tangent hyperplane to B in $\mathrm{PG}(n, q)$. Note that $\Omega \neq \langle Q, P \rangle$, since both are spread elements and cannot coincide since $\langle Q, P \rangle$ is an element of the blocking set, hence $\langle R, Q, P \rangle$ cannot be a tangent space.

The projection of $\langle R, \Omega \rangle$ from P onto $\mathrm{PG}(n, q)$ is an $(n-1)$-dimensional space through R in $\mathrm{PG}(n, q)$ which is skew to Q since $\Omega \cap \langle Q, P \rangle = \emptyset$, and which only has R in common with B since $\langle \Omega, R \rangle \cap \langle B, P \rangle = \{R\}$. Hence, this projection is a tangent $(n-1)$-space through R to B in $\mathrm{PG}(n, q)$. So we have shown that any tangent line in R to $\bar{B}$ gives rise to a tangent hyperplane to B in R. If any tangent line to $\bar{B}$ in R gives rise to a different tangent hyperplane to B, the theorem is proven.

Let η be a tangent hyperplane to B in R which is the projection of two tangent lines $\langle \Omega, R \rangle$ and $\langle \Omega', R \rangle$. The dimension of $\langle \eta, P \rangle$ is $2n - 3$, and $\dim(\langle \eta, P \rangle \cap W) = 2n - 4$. A hyperplane of $\mathrm{PG}(2n - 3, q)$ contains exactly one element of a regular $(n-2)$-spread. Since it contains Ω and Ω', $\Omega = \Omega'$. So η is the projection of at most one such $(n-1)$-space.

For every essential point Q of B, it is possible to select a tangent hyperplane H through Q, and to let this tangent hyperplane H play the role described in the preceding paragraph. Since Q is an affine essential point, this implies that Q lies in at least $q^{n-1} - aq^{n-2}$ tangent hyperplanes to B. $\qquad\square$

Lemma 8. *Let $\mathcal{S}$ be a partial $(q + a)$-cover of $\mathrm{PG}(n, q)$, $a < q$. If all holes of $\mathcal{S}$ are contained in a hyperplane π of $\mathrm{PG}(n, q)$, then $|\mathcal{H}_{\mathcal{S}}| \geq q^{n-1} - aq^{n-2}$.*

Proof. The hyperplanes of $\mathcal{S}$, together with the hyperplane π that contains all holes, form a cover of size $q + a + 1$, in which π is an essential hyperplane. Dualizing gives a blocking set B of size $q + a + 1$, where the dual of π is an essential point. Theorem 7 shows that the dual of π lies on at least $q^{n-1} - aq^{n-2}$ tangent hyperplanes to B. Dualizing again shows that π contains at least $q^{n-1} - aq^{n-2}$ points that are only covered by π. Removing π shows that there are at least $q^{n-1} - aq^{n-2}$ holes. $\qquad\square$

Remark 9. *The lower bound in Lemma 8 is sharp. Let $\mathcal{S}$ be the set of q hyperplanes through a fixed $(n-2)$-space π_{n-2}. Let H be the hyperplane through π_{n-2}, which is not chosen. Take a hyperplanes for which the $(n-2)$-dimensional intersections with H are all distinct and go through a common $(n-3)$-space of π_{n-2}, then there are exactly $q^{n-1} - aq^{n-2}$ holes.*

From now on, $\mathcal{S}$ denotes a partial $(q + a)$-cover of $\mathrm{PG}(n, q)$, $n \geq 3$. We denote the following property by (A_x):

(A_x) If $\mathcal{S}$ is a partial $(q + b)$-cover in $\mathrm{PG}(2, q)$, $b \leq x < (q - 2)/3$, with at most $q + b$ holes, then $q - b \leq |\mathcal{H}_{\mathcal{S}}| \leq q$ and the holes are collinear.

Note that we have shown in Theorem 5 and Lemma 8 that the property (A_x) holds for $x \leq (q - 10)/4$, $q > 13$.

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Lemma 10. *Assume (A_x) for all $x \leq a$. If a partial cover $\mathcal{S}'$ of $\mathrm{PG}(2, q)$ contains 3 non-collinear holes, then $|\mathcal{S}'| > q + a$.*

Proof. If $|\mathcal{S}'| = q + a$, this follows immediately from property (A_x). So suppose that $|\mathcal{S}'| = q + x'$, $x' < a$, and that there are 3 non-collinear holes, say H_1, H_2, H_3. Let P be a point not on $H_1 H_2$, $H_1 H_3$, or $H_2 H_3$. Adding $a - x'$ lines through P, different from PH_1, PH_2, PH_3, to the partial cover $\mathcal{S}'$ gives a cover $\mathcal{S}''$, with $|\mathcal{S}''| = q + a$. Applying property (A_x) to $\mathcal{S}''$, the corollary follows. $\square$

Lemma 11. *Assume (A_x) for all $x \leq a$, and $|\mathcal{H}_\mathcal{S}| \leq q^{n-1}$. A line that contains 2 holes of $\mathcal{S}$, contains at least $a + 3$ holes of $\mathcal{S}$.*

Proof. Let L be a line with t holes, $t < q - a$, and let π be a plane through L. Assumption (A_x) shows that if π contains at most $q + a$ holes, there are at least $q - a$ holes, which are all collinear, a contradiction. Hence, every plane through L contains at least $q + a + 1$ holes, which implies that there are at least

$$\theta_{n-2}(q + a + 1 - t) + t$$

holes in $\mathrm{PG}(n, q)$, which has to be at most q^{n-1}. If $t = a + 2$, $\theta_{n-2}(q + a + 1 - a - 2) + a + 2 > q^{n-1}$, a contradiction. Hence, t is at least $a + 3$. $\square$

Lemma 12. *Assume (A_x) for all $x \leq a$, and $|\mathcal{H}_\mathcal{S}| \leq q^{n-1}$. Every hole of $\mathcal{S}$ lies on more than $q^{n-2}/2$ lines with at least $q - a$ holes.*

Proof. Let R be a hole. There is a line L through R containing only covered points and R, otherwise there would be at least $\theta_{n-1} + 1$ holes. Using assumption (A_x) and Lemma 10, we see that a plane through L contains either at most $q - 1$ holes on a line through R, different from L, or it contains at least $q + x$ holes different from R.

Suppose that there are X planes through L with at most $q - 1$ holes different from R. Using assumption (A_x) and Lemma 10, we see that the number of holes is at least
$$X(q - a - 1) + (\theta_{n-2} - X)(q + a) + 1,$$

which has to be at most q^{n-1}. Putting $X = q^{n-2}/2$ yields a contradiction. Hence, there are more than $q^{n-2}/2$ planes with at most q holes. Again using assumption (A_x), we see that in each of these planes, there is a line through R containing at least $q - a - 1$ other holes, and all holes in such a plane lie on this line. $\square$

Theorem 13. *Assume (A_x) for all $x \leq a$, and $|\mathcal{H}_\mathcal{S}| \leq q^{n-1}$. Then the holes of $\mathcal{S}$ are contained in one hyperplane of $\mathrm{PG}(n, q)$.*

Proof. For $n = 2$, this is assumption (A_x) with $x = a$. Suppose by induction that this theorem holds for any dimension $i \leq n - 1$.

First, we show that there is a hyperplane π of $\mathrm{PG}(n, q)$ with at most q^{n-2} holes. Let R be a hole. There is a line L through R containing only covered points and R. Suppose that all planes through L contain more than q holes, then there would be at least $\theta_{n-2} q + 1$ holes, a contradiction. Suppose that there is a d-dimensional space π_d with at most q^{d-1} holes. Then there is a $(d + 1)$-dimensional space containing π_d with at most q^d holes. Otherwise, the

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number of holes would be at least $\theta_{n-d-1}(q^d + 1 - q^{d-1}) + q^{d-1}$, a contradiction if $d \leq n-1$. Hence, by induction, there is a hyperplane π of $\mathrm{PG}(n,q)$ with at most q^{n-2} holes.

Using the induction hypothesis, all holes in π are contained in an $(n-2)$-dimensional space π_{n-2} of π. Moreover, Lemma 8 shows that the number of holes in π_{n-2} is at least $q^{n-2} - aq^{n-3}$.

There are at least $\theta_{n-2}(q - a - 1) + 1$ holes in $\mathrm{PG}(n,q)$ since every plane through L contains at least $q - a - 1$ extra holes. Hence, there is certainly a hole R' that is not contained in π_{n-2}.

Now we distinguish between two cases.

Case 1: All lines through R' with at least $q-a$ holes are lines which intersect π_{n-2}. Lemma 12 shows that there are at least $q^{n-2}/2$ such lines. Since a line through two holes contains at least $a + 3$ holes (see Lemma 11), counting the holes in $\langle R', \pi_{n-2} \rangle$ yields that this number is at least

$$q^{n-2}(q - a - 1)/2 + (q^{n-2} - aq^{n-3} - q^{n-2}/2)(a + 2) + 1.$$

If all holes are contained in $\langle R', \pi_{n-2} \rangle$, the theorem is proven. Suppose now that not all holes are contained in the hyperplane $\langle R', \pi_{n-2} \rangle$. Let R'' be a hole not in $\langle R', \pi_{n-2} \rangle$. Connecting R'' with all the holes in $\langle R', \pi_{n-2} \rangle$ yields at least $(a+2)(q^{n-2}(q - a - 1)/2 + (q^{n-2} - aq^{n-3} - q^{n-2}/2)(a + 2) + 1) + 1$ holes, which is more than q^{n-1}, a contradiction.

Case 2: There is a line through R', skew to π_{n-2}, with at least $q-a$ holes. This yields at least

$$(q - a)(q^{n-2} - aq^{n-3})(a + 1) + q^{n-2} - aq^{n-3} + q - a > q^{n-1}$$

holes, a contradiction. $\qquad\square$

Theorem 14. *Assume (A_x) for all $x \leq a$, then the number of holes of $\mathcal{S}$ is at least $q^{n-1} - aq^{n-2}$.*

Proof. This follows immediately from Theorem 13 and Lemma 8. $\qquad\square$

Corollary 15. *Assume (A_x) for all $x \leq a$, and $|\mathcal{H}_{\mathcal{S}}| \leq q^{n-1}$. If q is a prime, $\mathcal{S}$ consists of q hyperplanes through a common $(n-2)$-space π and a other hyperplanes, not through π.*

Proof. It follows from Theorem 13 that the holes are contained in one hyperplane, say μ. Then the hyperplanes of $\mathcal{S}$, together with μ, constitute a cover $\mathcal{C}$ of size $q + a + 1 < 3(q + 1)/2$. Result 4, together with Result 2, shows that the unique minimal cover contained in $\mathcal{C}$ is the set of all hyperplanes through an $(n-2)$-space π. Since this set covers $\mathrm{PG}(n,q)$ entirely, the hyperplane μ is one of the hyperplanes through π. The other a hyperplanes are random, but do not contain π. $\qquad\square$

As remarked before, assumption (A_x) holds for all partial $(q + a)$-covers of $\mathrm{PG}(n,q)$, $x \leq (q - 10)/4$, $q > 13$.

However, a recent result of Blokhuis, Brouwer and Szőnyi [3] shows that assumption (A_x) is valid for all x, where $x < (q - 2)/3$. Moreover, the following example shows that the upper bound $a < (q - 2)/3$ is sharp.

Example 16. *Let $a = (q-2)/3$ and let $\mathcal{S}$ be a set of $q-1$ lines L_i through a point P, and $a+1$ other lines through a fixed point, lying on one of the lines L_i. Then there are $2(q-a-1) = q+a$ holes, lying on two lines.*

Combining Theorems 13, 14 and Corollary 15 with the result of Brouwer, Blokhuis and Szőnyi, yields the following theorem.

Theorem 17. *If $\mathcal{S}$ is a partial $(q+a)$-cover of $\mathrm{PG}(n,q)$, $a < (q-2)/3$, with at most q^{n-1} holes, then there are at least $q^{n-1} - aq^{n-2}$ holes and the holes are contained in one hyperplane. If q is a prime, $\mathcal{S}$ consists of q hyperplanes through a common $(n-2)$-space π and a other hyperplanes, not through π.*

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