Model order reduction with preservation of passivity, non-expansivity and Markov moments

L. Knockaert1, T. Dhaene, F. Ferranti, D. De Zutter
Dept. Information Technology, Ghent University
Interdisciplinary Institute for Broadband Technology
Gaston Crommenlaan 8, PB 201, B-9050 Gent, Belgium.

Abstract
A new model order reduction (MOR) technique is presented which preserves passivity and non-expansivity. It is a projection-based method which exploits the solution of Linear Matrix Inequalities (LMI’s) to generate a descriptor state space format which preserves positive-realness and bounded-realness. In the case of both non-singular and singular systems, solving the LMI can be replaced by equivalently solving an algebraic Riccati equation (ARE), which is known to be a more efficient approach. A new ARE and a frequency inversion technique are also presented to specifically deal with the important singular case. The preservation of Markov moments is also guaranteed by the judicious choice of a projection matrix.

Key words: Passivity, non-expansivity, positive-real lemma, bounded-real lemma, model order reduction
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1. INTRODUCTION

The use of model order reduction (MOR) aiming at obtaining compact descriptions of initially large linear state space models has become a standard component in computer-aided design methodologies for a large number of engineering and physics applications. For a good introductory textbook on MOR the reader is referred to [1]. Three MOR approaches can currently be distinguished [2]. The first approach consists of the SVD-based methods, comprising the balanced realization method [3] and Hankel norm approximation [4]. The second approach consists of the projection-based Krylov-subspace methods [5], comprising the Laguerre-SVD approach [6, 7]. The third approach consists of iterative methods combining aspects of both the SVD and Krylov methods [8]. In the excellent overview paper [2] both strengths and weaknesses of the three approaches are analyzed; e.g., the first and third approaches generally preserve stability, while the second approach is fast but does not in general guarantee stability (but see also [7]).

Passivity is an important property to satisfy because stable, but non-passive macro-models can produce unstable systems when connected to other stable, even passive, loads. It is well-known that passivity is equivalent with the positive-realness of the system transfer function. The equivalent form of passivity for a scattering matrix representation is non-expansivity or bounded-realness [9, 10]. It is well established that model reduction techniques with preservation of passivity mostly belong to the balanced truncation class [11–13] or are spectral interpolation-based methods [14–16]. In the case of projection-based Krylov methods the problem of preservation of passivity has been studied by several researchers; for an overview of existing approaches see [6, 17–21]. The problem with the Krylov-based passivity preserving methods is that they...
often assume a special descriptor state space setting that may not always be feasible [11]. In this paper, we present a new passivity-preserving and non-expansivity-preserving MOR technique, which does not require any special internal structure of the state space model. It is a projection-based method which exploits the solution of Linear Matrix Inequalities (LMI’s) to generate a descriptor state space format which preserves positive-realness and bounded-realness. In the case of both non-singular and singular systems, solving the LMI can be replaced by equivalently solving an algebraic Riccati equation (ARE), which is known to be a more efficient approach [22].

The paper is organized as follows. Section 2 describes the new technique and contains the proof of its passivity-preserving and non-expansivity-preserving properties. Section 3 deals with the important singular case and presents a new ARE and a frequency inversion technique specifically tailored to the singular case. Finally, Section 4 presents pertinent choices for the Krylov projection matrices in such a way that the Markov moments of the system are also preserved.

2. MAIN RESULTS

Notation: Throughout the paper $X^T$ and $X^H$ respectively denote the transpose and Hermitian transpose of a matrix $X$, and $I_n$ denotes the identity matrix of dimension $n$. For two Hermitian matrices $X$ and $Y$, the matrix inequalities $X > Y$ or $X \geq Y$ mean that $X - Y$ is respectively positive definite or positive semidefinite. Of course, $X < Y$ or $X \leq Y$ means $Y > X$ or $Y \geq X$. The closed right halfplane $\Re [s] \geq 0$ is denoted $C_+$.

2.1. Positive-real systems

For the real system with minimal realization

\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}

where $B \neq 0$, $C \neq 0$ are respectively $n \times p$ and $p \times n$ real matrices and $A \neq 0$ is a $n \times n$ real matrix, to be passive, it is required that the $p \times p$ transfer function

$$H(s) = C(sI_n - A)^{-1}B + D$$

is analytic in $C_+$, such that

$$H(s) + H(s)^H \geq 0 \quad \forall s \in C_+$$

It is well-known [9] that the positive-real lemma in Linear Matrix Inequality (LMI) format: \( \exists \ P^T = P > 0 \)

such that

$$
\begin{bmatrix}
A^T P + PA & PB - CT \\
B^T P - C & -D - DT
\end{bmatrix} \leq 0
$$

(2)

guarantees the passivity of the system (1). With the additional stronger condition $D + DT > 0$ (strict passivity at $s = \infty$), the LMI (2) is feasible if and only if there exists a real matrix $P^T = P > 0$ satisfying the algebraic Riccati equation (ARE)

$$A^T P + PA + (PB - CT)W_p(PB - CT)^T = 0$$

(3)

where

$$W_p = (D + DT)^{-1}$$

The ARE (3) is generally solved by constructing the associated Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix}
A - BW_pC & BW_pB^T \\
-CTW_pC & -A^T + CTW_pB^T
\end{bmatrix}$$

(4)
Then the system (1) is passive, i.e., the LMI (2) is feasible, if and only if $\mathcal{H}$ has no purely imaginary eigenvalues [23].

Before tackling the main results, we need to define what is meant by a descriptor state space system. It is a more general system described by the differential equations

\begin{align*}
E \dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}

(5a) (5b)

where $E \neq 0$ is a $n \times n$ real matrix called the descriptor. In descriptor state space format the transfer function is given by

$$H(s) = C(sE - A)^{-1}B + D$$

Note that it is usually required that $sE - A$ is a regular matrix pencil, i.e., $\det(sE - A) = 0$ has a finite number of $s$ values as solutions. In our case we will only need the simple nonsingular descriptor state space format with $E$ nonsingular.

Next suppose $H(s)$ is passive. The following theorem provides a means to obtain a reduced model which preserves passivity.

**Theorem 2.1.** Suppose the system (1) is passive and let $P = P^T > 0$ be a solution of the LMI (2). Let $U$ be a $n \times r$, $1 \leq r \leq n$ matrix of full rank. Then the reduced descriptor state space system with transfer function

$$H_1(s) = CU(sU^T P U - U^T P A U)^{-1} U^T P B + D$$

is passive.

**Proof.** It is clear that $H_1(s)$ can be written as

$$H_1(s) = \tilde{C}(sI_r - \tilde{A})^{-1}\tilde{B} + D$$

where

\begin{align*}
\tilde{A} &= (U^T P U)^{-1} U^T P A U \\
\tilde{C} &= CU \\
\tilde{B} &= (U^T P U)^{-1} U^T P B
\end{align*}

Putting $\tilde{P} = U^T P U$, it is clear that $\tilde{P}^T = \tilde{P} > 0$. Next consider the matrix

$$L_1 = \begin{bmatrix}
\tilde{A}^T \tilde{P} + \tilde{P} \tilde{A} & \tilde{P} \tilde{B} - \tilde{C}^T \\
\tilde{B}^T \tilde{P} - \tilde{C} & -D - D^T
\end{bmatrix}$$

Putting $\tilde{P} = U^T P U$, it is clear that $\tilde{P}^T = \tilde{P} > 0$. Next consider the matrix

$$L_1 = \begin{bmatrix}
\tilde{A}^T \tilde{P} + \tilde{P} \tilde{A} & \tilde{P} \tilde{B} - \tilde{C}^T \\
\tilde{B}^T \tilde{P} - \tilde{C} & -D - D^T
\end{bmatrix}$$

(6)

It is easy to show that the matrix $L_1$ can be written as

$$L_1 = \mathcal{E}^T \begin{bmatrix}
A \dot{P} + PA & PB - C^T \\
B^T \dot{P} - C & -D - D^T
\end{bmatrix} \mathcal{E}$$

where

$$\mathcal{E} = \begin{bmatrix}
U & 0_{n \times p} \\
0_{p \times n} & I_p
\end{bmatrix}$$

By virtue of the LMI (2) we conclude that $L_1 \leq 0$ and the proof is complete.
2.2. Bounded-real systems

For the real system with minimal realization (1) to be non-expansive, it is required that the transfer function $H(s)$ is analytic in $\mathbb{C}_+$ such that

$$H(s)H(s) \leq I_p \quad \forall s \in \mathbb{C}_+$$

In this case (see [9]), it is well-known that the bounded-real lemma in LMI format: \( \exists P^T = P > 0 \) such that

$$
\begin{bmatrix}
    A^TP + PA + C^TC & PB + C^TD \\
    B^TP + D^TC & D^TD - I_p
\end{bmatrix} \leq 0
$$

guarantees the non-expansivity of the system (1). With the additional stronger product condition $D^TD < I_p$ (strict non-expansivity at $s = \infty$), the LMI (7) is feasible if and only if there exists a real matrix $P^T = P > 0$ satisfying the ARE

$$
A^TP + PA + C^TC + (PB + C^TD)W_s(PB + C^TD)^T = 0
$$

where

$$W_s = (I_p - D^TD)^{-1}$$

The ARE (8) is solved by constructing the associated Hamiltonian matrix

$$
\tilde{H} = \begin{bmatrix}
    A + BW_sD^TC & BW_sB^T \\
    -C^TW_sC & -A^T - C^TDW_sB^T
\end{bmatrix}
$$

Then the system (1) is non-expansive, i.e., the LMI (7) is feasible if and only if $\tilde{H}$ has no purely imaginary eigenvalues [23].

Suppose $H(s)$ is non-expansive. The following theorem provides a means to obtain a reduced model which preserves non-expansivity.

**Theorem 2.2.** Suppose the system (1) is non-expansive and let $P = P^T > 0$ be a solution of the LMI (7). Let $U$ be a $n \times r$, $1 \leq r \leq n$ matrix of full rank. Then the reduced descriptor state space system with transfer function

$$H_2(s) = CU(sU^TPU - U^TPAU)^{-1}U^TPB + D$$

is non-expansive.

**Proof.** Similar as Theorem 2.1. It is clear that $H_2(s)$ can be written as

$$H_2(s) = \tilde{C}(sI_r - \tilde{A})^{-1}\tilde{B} + D$$

where

$$\tilde{A} = (U^TPU)^{-1}U^TPAU$$

$$\tilde{C} = CU \quad \tilde{B} = (U^TPU)^{-1}U^TPB$$

Putting $\tilde{P} = U^TPU$, it is clear that $\tilde{P}^T = \tilde{P} > 0$. Next consider the matrix

$$
\mathcal{L}_2 = \begin{bmatrix}
\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} + \tilde{C}^T\tilde{C} & \tilde{P}B + \tilde{C}^TD \\
B^T\tilde{P} + D^T\tilde{C} & D^TD - I_p
\end{bmatrix}
= \begin{bmatrix}
U^T(A^TP + PA + C^TC)U & U^T(PB + C^TD) \\
(B^TP + D^TC)U & D^TD - I_p
\end{bmatrix}
$$

It is easy to show that the matrix $\mathcal{L}_2$ can be written as

$$\mathcal{L}_2 = \mathcal{E}^T \begin{bmatrix}
A^TP + PA + C^TC & PB + C^TD \\
B^TP + D^TC & D^TD - I_p
\end{bmatrix} \mathcal{E}$$

with $\mathcal{E}$ as in (6). By virtue of the LMI (7) we conclude that $\mathcal{L}_2 \leq 0$ and the proof is complete.
2.3. Markov moment preservation

In Section 4 we will show how the projection matrix $U$ can be chosen in order to preserve a selection of the so-called Markov moments of the system.

3. THE SINGULAR CASE

In the positive-real case the LMI (2) and the ARE (3) are equivalent only in the case $W_p > 0$ or $D + D^T > 0$. Similarly, in the bounded-real case the LMI (7) and the ARE (8) are equivalent only in the case $W_s > 0$ or $D^T D < I_p$. It is seen that the singular cases $D + D^T$ singular or $I_p - D^T D$ singular cannot easily be solved by means of ARE’s (but see also [24] and [25] for that matter), since the pertinent Hamiltonian matrices are then undefined. On the other hand, LMI’s are convex formulations and can always be solved by convex optimization [26], without needing ARE solvers and/or Hamiltonian matrices. However, we will show we can say more under sufficiently general conditions and still use the ARE formalism. Our approach differs considerably from the approaches in [24] and [25] in that in our method no state space transformations are needed to obtain the ARE’s for the singular case. In order to concentrate solely on the positive-real case we first state the following equivalence Lemmas relating bounded-real and positive-real cases:

**Lemma 3.1.** $H(s) = C(sI_n - A)^{-1}B + D$ minimal and bounded-real with $A$ Hurwitz is equivalent with $G(s) = C_1(sI_n - A)^{-1}B + D_1$ positive-real for $C_1, D_1$ as constructed below.

**Proof.** We have

$$I_p - H^T(-s)H(s) \geq 0 \quad \forall s \in \mathbb{C}_+$$

Now

$$I_p - H^T(-s)H(s) = I - D^T D - (D^T C + B^T W_c (sI_n - A)^{-1} B - B^T (-sI_n - A^T)^{-1} (C^T D + W_c B)$$

where $W_c > 0$ is the controllability Grammian.

Hence taking $D_1 = (I - D^T D)/2$ and $C_1 = -D^T C - B^T W_c$ we see that

$$G^T(-s) + G(s) \geq 0 \quad \forall s \in \mathbb{C}_+$$

and the proof is complete.

**Lemma 3.2.** If $H(s) = C(sI_n - A)^{-1}B + D$ is minimal and bounded-real such that $\det[I_p - H(s)] \neq 0$ for $\Re[s] > 0$ then $G(s) = [I_p - H(s)]^{-1}[I_p + H(s)] = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}$ is minimal and positive-real with

$$\hat{A} = A + B(I_p - D)^{-1} C, \quad \hat{B} = \sqrt{2} B(I_p - D)^{-1}$$

$$\hat{C} = \sqrt{2} (I_p - D)^{-1} C, \quad \hat{D} = (I_p - D)^{-1} (I_p + D)$$

Conversely, if $G(s) = C(sI_n - A)^{-1}B + D$ is minimal and positive-real then $H(s) = [G(s) - I_p][G(s) + I_p]^{-1} = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}$ is minimal and bounded-real with

$$\hat{A} = A - B(I_p + D)^{-1} C, \quad \hat{B} = \sqrt{2} B(I_p + D)^{-1}$$

$$\hat{C} = \sqrt{2} (I_p + D)^{-1} C, \quad \hat{D} = (I_p - D)(I_p + D)^{-1}$$

**Proof.** See [27].

In order to proceed in the singular positive-real case, we first need two more Lemmas:
Lemma 3.3. If \( D + D^T \geq 0 \) and \( \text{rank}(D + D^T) = r < p \) there exists a \( p \times p \) orthogonal transformation matrix \( \Gamma \) such that
\[
\Gamma^T (D + D^T) \Gamma = \begin{bmatrix} R_r & 0 \\ 0 & 0 \end{bmatrix}
\]
where the \( r \times r \) matrix \( R_r \) is symmetric positive definite. The positive-realness of \( \tilde{H}(s) = \Gamma^T H(s) \Gamma \) is not affected by this transformation.

Proof. See [24]. Note that \( r = 0 \) corresponds to the totally singular case \( D + D^T = 0 \).

Lemma 3.4. Suppose \( B \) and \( C \) are full rank. Then there exists a matrix \( P = P^T > 0 \) that satisfies \( PB = C^T \) if and only if \( CB = B^T C^T > 0 \). Furthermore, in that case, all positive definite solutions of \( PB = C^T \) are given by
\[
P = C^T (CB)^{-1} C + B_\perp X B_\perp^T
\]
where \( X \) is an arbitrary \((n-p) \times (n-p)\) positive definite matrix and \( B_\perp \) is the orthonormal null space of \( B \).

Proof. See [28]. Note that if \( \ker(B) = \{0\} \), which can happen when \( p \geq n \), the only solution is \( P = C^T (CB)^{-1} C \).

The next theorem provides an ARE approach for the singular positive-real case.

Theorem 3.1. Suppose the positive-real singular system is as in Lemma 3.3, i.e.,
\[
D + D^T = \begin{bmatrix} R_r & 0 \\ 0 & 0 \end{bmatrix}
\]
with \( R_r \) positive definite. Then provided the matrices
\[
C_s B_s
\]
and
\[
\mathcal{R} = -(C_s A B_s)^T - C_s A B_s - (C_s B_r - B_r^T C_r^T) R_r^{-1} (B_r^T C_r^T - C_r B_s)
\]
(constuctively defined in the proof below) are symmetric positive definite, there is a positive definite solution \( P \) of the composite algebraic Riccati equation (constructively defined in the proof below):
\[
A^T P + P A + (P B_r - C_r^T) R_r^{-1} (P B_r - C_r^T)^T + (P B - C^T) R^{-1} (P B - C^T)^T = 0 \tag{10}
\]

Proof. We start with the LMI formulation by means of the Lur’e equations [29]:
\[
A^T P + P A = -Q^T Q \\
P B - C^T = -Q^T W \\
D + D^T = W^T W
\]

Partitioning the matrices \( B, C, Q \) and \( W \) as
\[
B = [B_r, B_s] \quad C = [C_r^T, C_s^T]^T \quad Q = [Q_r^T, Q_s^T]^T \quad W = \begin{bmatrix} W_r & 0 \\ 0 & 0 \end{bmatrix}
\]
we can reformulate the Lur’e equations as:
\[
A^T P + P A = -Q_r^T Q_r - Q_s^T Q_s \tag{11a} \\
P B_r - C_r^T = -Q_r^T W_r \tag{11b} \\
P B_s - C_s^T = 0 \tag{11c} \\
R_r = W_r^T W_r \tag{11d}
\]
Eliminating equations (11d) and (11b) we obtain
\[ A^T P + PA + (PB_r - C_r^T)R_r^{-1}(PB_r - C_r^T)^T = -Q_s^T Q_s \quad (12a) \]
\[ PB_r - C_r^T = 0 \quad (12b) \]

If the aim were solely to solve equation (12b), we could utilize Lemma 3.4, but in general this will not be sufficient (except when \( \ker(B_s) = \{0\} \)), since we also need to satisfy equation (12a). Anyway, a first necessary condition for the existence of a positive definite \( P \) is \( C_s B_s = (C_s B_s)^T > 0 \) (see also [24]). Next, if we right-multiply equation (12a) with \( B_s \), we obtain
\[ A^T C_s^T + PAB_s + (PB_r - C_r^T)R_r^{-1}(B_r^T C_s^T - C_r B_s) = -Q_s^T Q_s B_s \quad (13) \]

Defining \( W_s = Q_s B_s \) and left-multiplying equation (13) with \( B_s^T \), we obtain
\[ (C_s A B_s)^T + C_s A B_s + (C_s B_r - B_s^T C_r^T)R_r^{-1}(B_r^T C_s^T - C_r B_s) = -W_s W_s \quad (14) \]

Defining
\[ V = B_r^T C_s^T - C_r B_s \]
\[ B = AB_s + B_r R_r^{-1} V \]
\[ C = -C_s A + V^T R_r^{-1} C_r \]
\[ R = -(C_s A B_s)^T - C_s A B_s - V^T R_r^{-1} V \]

we can rewrite equations (13) and (14) as
\[ P B - C^T = -Q_s^T W_s \]
\[ R = W_s W_s \]

Assuming \( R \) positive definite, we can write
\[ Q_s^T Q_s = (P B - C^T) R^{-1} (P B - C^T)^T \]
yielding the following composite algebraic Riccati equation for \( P \):
\[ A^T P + PA + (PB_r - C_r^T)R_r^{-1}(PB_r - C_r^T)^T + (P B - C^T)R^{-1}(P B - C^T)^T = 0 \]
and the proof is complete.

Remark: in the totally singular case \( D + D^T = 0 \) the Riccati equation becomes
\[ A^T P + PA + (P B - C^T)R^{-1}(P B - C^T)^T = 0 \]
with
\[ B = AB \]
\[ C = -CA \]
\[ R = -(CAB)^T - CAB \]

As a last result, which can also help finding the LMI matrix \( P \) in the singular case, we have the following:

**Theorem 3.2.** Frequency inversion theorem: Let \( H(s) = C(sI_n - A)^{-1}B + D \) be minimal and positive-real with \( A \) Hurwitz. Then \( G(s) = C(sI_n - A)^{-1}\tilde{B} + \tilde{D} \) with
\[ \tilde{A} = A^{-1} \]
\[ \tilde{B} = A^{-1} B \]
\[ \tilde{C} = -CA^{-1} \]
\[ \tilde{D} = D - CA^{-1} B \]
is also positive real and admits the same \( P \) matrix as \( H(s) \).
**Proof.** It is straightforward to see that when $A$ is Hurwitz, then $A^{-1}$ is also Hurwitz and vice versa. Also, it is simple to see by substitution (see also [30]) that $G(s) = H(1/s)$. By positive-realness, $H(s)$ admits a factorization [29]:

$$H(s) + H(-s)^T = M(-s)^T M(s) \quad \forall s \in \mathbb{C}_+$$

Since the mapping $s \mapsto 1/s$ is one-to-one in (extended) $\mathbb{C}_+$, it follows that

$$G(s) + G(-s)^T = H(1/s) + H(-1/s)^T = M(-1/s)^T M(1/s) \quad \forall s \in \mathbb{C}_+$$

In other words $G(s)$ is positive-real. To prove it admits the same $P$ as $H(s)$ we write the Lur’e equations

$$A^T P + PA = -Q^T Q$$
$$PB - CT = -Q^T W$$
$$D + DT = W^T W$$

Define $Q = -QA^{-1}$ and $W = W - QA^{-1}B$. It is easy to see that

$$\tilde{A}^T P + P \tilde{A} = -Q^T Q$$

Also

$$-Q^T W = A^{-T} [Q^T W - Q^T QA^{-1}B] = P \tilde{B} - \tilde{C}^T$$

and finally

$$\tilde{D} + \tilde{D}^T = W^T W$$

which completes the proof.

Note that $\tilde{D} = H(0)$ and hence Theorem 3.2 maps the positive-realness problem from $s = \infty$ to $s = 0$. Of course it could be that both $H(\infty) + H(\infty)^T$ and $H(0) + H(0)^T$ are singular, in which case Theorem 3.1 or the approaches in [24] and [25] will provide solutions.

**4. Markov Moment Preservation**

In the Section 2 we showed that passivity and non-expansivity can be preserved by introducing a full rank matrix $U$. In this section we will show how pertinent column-orthogonal projection matrices $U$ can be constructed which also preserve the so-called Markov moments of the system. To see this, we first write the Laurent expansion of

$$H(s) = C(sI_n - A)^{-1}B + D = C(sP + G)^{-1}R + D$$

with $G = -PA, R = PB$, in the vicinity of $s = \infty$.

We have

$$H(s) = D + \sum_{k=0}^{\infty} (-1)^k s^{-k-1} C \Omega^k B$$

where $\Omega = -A$. This can be written as

$$H(s) = \sum_{k=-1}^{\infty} (-1)^k s^{-k-1} M_k$$

The coefficients $M_k = C \Omega^k B, k \geq 0$ and $M_{-1} = -D$ are known (up to a sign) as the Markov moments of $H(s)$ at $s = \infty$. Next consider the $n \times r$ Krylov matrix ($r = pq \leq n$)

$$K = [B, \Omega B, \Omega^2 B, \ldots, \Omega^{r-1} B]$$
and consider choosing an orthonormal basis for the columns of $K$, which can be implemented by performing the 'thin' SVD of the Krylov matrix as $K = U \Sigma V^T$, and where the $n \times r$ matrix $U$ is column-orthogonal.

Putting

$$\tilde{P} = U^T P \quad \tilde{G} = U^T G U \quad \tilde{R} = U^T R$$

$$\tilde{C} = C U \quad \tilde{\Omega} = P^{-1} \tilde{G} \quad \tilde{B} = P^{-1} \tilde{R}$$

the new Markov moments are given by

$$\tilde{M}_{k}^{-1} = M_{k}^{-1} = -D \quad \tilde{M}_{k} = \tilde{C} \tilde{\Omega}^k \tilde{B} \quad k = 0, 1, \ldots$$

We are now in a position to prove (see also [31]):

**Theorem 4.1.** With the choice of $U$ as above, the Markov moments are equal up to order $q - 1$, i.e., $\tilde{M}_{k} = M_{k}$ for $k = 0, 1, \ldots, q - 1$.

**Proof.** Since we have constructed an orthonormal basis for the columns of $K$, we can write $\Omega^k B = UW_k$, $k = 0, \ldots, q - 1$, where $W_k$ is a $r \times p$ matrix. Note that we have $R = PB = PUW_0$ and $\tilde{R} = U^T R = U^T PUW_0 = \tilde{P}W_0$ and hence $\tilde{B} = \tilde{P}^{-1} \tilde{R} = W_0$. Next consider the $n \times n$ matrix

$$Z = U \tilde{P}^{-1} U^T G$$

By induction, it is easy to prove that $Z^k U = U \tilde{\Omega}^k$ for $k = 0, \ldots, q - 1$ and hence

$$\tilde{M}_{k} = \tilde{C} \tilde{\Omega}^k \tilde{B} = CZ^k UW_0 = CZ^k B \quad k = 0, \ldots, q - 1$$

There remains to prove that $Z^k B = \Omega^k B$ for $k = 0, \ldots, q - 1$. This is clearly the case for $k = 0$. Next suppose that $Z^k B = \Omega^k B$ for some $k$. Then

$$P^{-1} GZ^k B = \Omega^{k+1} B = UW_{k+1}$$

Pre-multiplying by $U^T P$ yields

$$U^T GZ^k B = U^T PUW_{k+1} = \tilde{P}W_{k+1}$$

or

$$W_{k+1} = \tilde{P}^{-1} U^T GZ^k B$$

and hence

$$Z^{k+1} B = U \tilde{P}^{-1} U^T GZ^k B = UW_{k+1} = \Omega^{k+1} B$$

which completes the proof.

Recall that by Theorems 2.1 and 2.2, the reduced order model is passive resp. non-expansive, when the original transfer function $H(s)$ is passive resp. non-expansive. Also, one often wishes to have equal Markov moments calculated about another point than infinity, or else to have Markov moments which are coefficients of a Laguerre expansion [6, 7]. All these possibilities can be dealt with by transforming the Laplace variable $s$ by means of a real Möbius transformation

$$s = \frac{\alpha u + \beta}{\gamma u + \delta} \quad \alpha \delta - \beta \gamma \neq 0 \quad (15)$$

The resulting transfer function in the $u$-domain is

$$(\gamma u + \delta)C [u(\alpha P + \gamma G) + (\beta P + \delta G)]^{-1} R + D$$

Now assuming that $\alpha P + \gamma G$ is nonsingular, we can define the matrices

$$\tilde{B} = (\alpha P + \gamma G)^{-1} R \quad \tilde{\Omega} = (\alpha P + \gamma G)^{-1} (\beta P + \delta G)$$

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After construction of a base \( \hat{U} \) of the Krylov matrix

\[ \hat{\mathcal{K}} = \left[ \hat{B}, \hat{\Omega}\hat{B}, \hat{\Omega}^2\hat{B}, \ldots, \hat{\Omega}^{q-1}\hat{B} \right] = \hat{U}\hat{\Sigma}\hat{V}^T \]

the reduced matrices are now

\[ \hat{P} = \hat{U}^T \hat{P}\hat{U} \quad \hat{G} = \hat{U}^T \hat{G}\hat{U} \quad \hat{R} = \hat{U}^T \hat{R} \quad \hat{C} = \hat{C}\hat{U} \]

For example, inserting \( \alpha = s_0, \beta = \gamma = 1, \delta = 0 \) in (15), we in fact perform a Taylor expansion about \( s_0 \), as in [32], and inserting \( \beta = \alpha, \gamma = -1, \delta = 1 \) in (15), boils down to a scaled Laguerre expansion with scaling factor \( \alpha > 0 \), as in [6, 7]. Of course, by Theorems 2.1 and 2.2, passivity and non-expansivity are always maintained.

5. CONCLUSION

We have presented a new model order reduction technique which preserves passivity and non-expansivity. It is a projection-based method which exploits the solution of Linear Matrix Inequalities to generate a descriptor state space format which preserves positive-realness and bounded-realness. In the case of both non-singular and singular systems, solving the LMI can be replaced by equivalently solving an algebraic Riccati equation, which is known to be a faster approach. A new ARE and a frequency inversion technique are presented to specifically deal with the difficult singular case. Last but not least, we have shown how the pertinent column-orthogonal projection matrix can be constructed such that the Markov moments of the system are also preserved.

References