The polynomial null solutions of a higher spin Dirac operator in two vector variables

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Abstract

The polynomial null solutions are studied of the higher spin Dirac operator $\mathcal{Q}_{k,l}$ acting on functions taking values in an irreducible representation space for $\text{Spin}(m)$ with highest weight $(k + \frac{1}{2}, l + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$, with $k, l \in \mathbb{N}$ and $k \geq l$.

Keywords: Clifford analysis, Dirac operators, higher spin

2000 Mathematics Subject Classification: 30G35, 22E46, 15A66

1 Introduction

Consider an oriented spin manifold, i.e. a Riemannian manifold with a spin structure which allows the construction of vector bundles whose underlying symmetry group is $\text{Spin}(m)$ rather than $SO(m)$, see e.g. [21]. On such a Riemannian spin manifold there is a whole system of conformally invariant, elliptic, first-order differential operators acting on sections of an appropriate spin bundle, whose existence and construction can be established through geometrical and representation theoretical methods, see e.g. [6, 15, 25, 26]. In Clifford analysis these operators are studied from a function theoretical point of view, mainly focusing on their rotational invariance with respect to the spin group $\text{Spin}(m)$, or its Lie algebra $\mathfrak{so}(m)$, and considering functions on $\mathbb{R}^m$ instead of sections. The simplest example is the Dirac operator, acting on spinor-valued functions; we refer to the standard references [1, 13, 17]. Next in line are the Rarita-Schwinger operator, acting on functions with values in the irreducible $\text{Spin}(m)$ -representation with highest weight $(\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2})$, and its generalizations, denoted $\mathcal{R}_k$ with $k \in \mathbb{N}$, to the case of functions taking values in irreducible representation spaces with highest weight $(k + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$, see e.g. [9, 10]. Also higher spin Dirac operators acting on spinor-valued forms have been studied in detail, see e.g. [8, 24]. In [2] we constructed the (unique up to a multiplicative constant) higher spin Dirac operator $\mathcal{Q}_{k,l}$ acting between functions taking values in an irreducible $\text{Spin}(m)$ -representation with highest weight $(k + \frac{1}{2}, l + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$, with $k, l \in \mathbb{N}$ and $k \geq l$.

Higher spin operators behave completely different from the classical Dirac operator with respect to their polynomial solutions: spaces of these solutions are no longer irreducible as a $\text{Spin}(m)$ -module. A typical problem is to decompose them into $\text{Spin}(m)$ -irreducibles. A detailed description of the homogeneous polynomial null solutions of the Rarita-Schwinger operator $\mathcal{R}_k$ is given in [9, 10]. In this paper the decomposition of the polynomial kernel space for $\mathcal{Q}_{k,l}$ into $\text{Spin}(m)$ -irreducibles is studied; it requires a different approach than the Rarita-Schwinger case.

2 Clifford analysis, definitions and propositions

Let $(\underline{e}_1, \dots, \underline{e}_m)$ be an orthonormal basis for the Euclidean space $\mathbb{R}^m$. We denote by $\mathbb{C}_m$ the complex universal Clifford algebra, generated by these basis elements, its multiplication being governed by the relations $\underline{e}_i \underline{e}_j + \underline{e}_j \underline{e}_i = -2\delta_{ij}$, $i, j = 1, \dots, m$. The space $\mathbb{R}^m$ is embedded in $\mathbb{C}_m$ by identifying $(x_1, \dots, x_m)$ with the real Clifford vector $\underline{x} = \sum_{j=1}^m \underline{e}_j x_j$. The multiplication of two vectors $\underline{x}$ and $\underline{y}$ is given by $\underline{x}\underline{y} = -\langle \underline{x}, \underline{y} \rangle + \underline{x} \wedge \underline{y}$ with

$$\langle \underline{x}, \underline{y} \rangle = \sum_{j=1}^m x_j y_j \quad \text{and} \quad \underline{x} \wedge \underline{y} = \sum_{1 \leq i < j \leq m} \underline{e}_i \underline{e}_j (x_i y_j - x_j y_i)$$

the scalar-valued Euclidean inner product and the bivector-valued wedge product respectively.

For convenience, we will work in odd dimension $m = 2n + 1$. In this case there is a unique spinor space $\mathbb{S}$, as opposed to the even-dimensional case $m = 2n$ where there are two spinor representations (often referred to as even and odd spinors). However, these cases do not differ from each other conceptually: in case of even dimensions $m = 2n$, it suffices to take into account that the vector-valued (higher spin) Dirac operator will change the parity of the underlying values. The spinor space $\mathbb{S}$ should be thought of as a minimal left ideal in $\mathbb{C}_m$, which can be defined in terms of a primitive idempotent; it is characterized by the highest weight $(\frac{1}{2}, \dots, \frac{1}{2})$ under the standard multiplicative action of the spin group

$$\text{Spin}(m) = \left\{ \prod_{j=1}^{2k} s_j : k \in \mathbb{N}, s_j \in S^{m-1} \right\},$$

with S^{m-1} the unit sphere in $\mathbb{R}^m$. In case one prefers working with its Lie algebra $\mathfrak{so}(m)$, which can be identified with the subspace of bivectors in the algebra $\mathbb{C}_m$, the derived action should be used.

The Dirac operator is denoted $\underline{\partial}_x = \sum_{j=1}^m \underline{e}_j \partial_{x_j}$. It is an elliptic $\text{Spin}(m)$ -invariant first-order differential operator acting on spinor-valued functions $f(\underline{x})$ on $\mathbb{R}^m$. It factorizes the Laplace operator: $\Delta_x = -\underline{\partial}_x^2$ on $\mathbb{R}^m$. An $\mathbb{S}$ -valued function f is monogenic in an open region $\Omega \subset \mathbb{R}^m$ if and only if it satisfies $\underline{\partial}_x f = 0$ in Ω . A monogenic homogeneous polynomial is often termed a spherical monogenic for short. For a detailed account

of the theory of monogenic functions, so called Euclidean Clifford analysis, we refer the reader to e.g. [1, 13, 17]. In particular, we mention the Cauchy-Kowalewskaia (CK) extension principle. Considering $\mathbb{R}^{m-1}$ as the hyperplane $x_m = 0$ in $\mathbb{R}^m$, a real-analytic function $f^*(\underline{x}^*)$, $\underline{x}^* = \sum_{j=1}^{m-1} e_j x_j$ in $\mathbb{R}^{m-1}$ can be uniquely extended to a monogenic function $f(\underline{x}^*, x_m)$ in an open neighbourhood in $\mathbb{R}^m$ of $\mathbb{R}^{m-1}$, this so-called CK-extension of $f^*(\underline{x}^*)$ being given by $f(\underline{x}^*, x_m) = \exp(-x_m \underline{\partial}_x^*) f^*(\underline{x}^*)$ with $\underline{\partial}_x^* = \sum_{i=1}^{m-1} e_i \partial_{x_i}$.

Some notational remarks. We will often need to refer to the highest weight of a representation; to that end we introduce the short notation $(\lambda_1, \dots, \lambda_N)$ for $(\lambda_1, \dots, \lambda_N, 0, \dots, 0)$ and denote by $(\lambda_1, \dots, \lambda_N)'$ the highest weight $(\lambda_1 + \frac{1}{2}, \dots, \lambda_N + \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. Further, $N \in \mathbb{N}$ and $\underline{\partial}_i$ is short for the Dirac operator ∂_{u_i} . By $\text{Ker}_h \mathcal{D}$ we denote the vector space of h -homogeneous polynomial solutions of the operator $\mathcal{D}$.

In [12, 17] it was shown that irreducible (finite-dimensional) $\text{Spin}(m)$ -modules of integer (resp. half-integer) weight can be realized in terms of harmonic (resp. monogenic) polynomials of several vector variables.

Definition 1. A function $f : \mathbb{R}^{Nm} \rightarrow \mathbb{C}$, $(\underline{u}_1, \dots, \underline{u}_N) \mapsto f(\underline{u}_1, \dots, \underline{u}_N)$ is *simplicial harmonic* if the following conditions are satisfied:

$$\begin{aligned} \langle \underline{\partial}_i, \underline{\partial}_j \rangle f &= 0, & i, j &= 1, \dots, N \\ \langle \underline{u}_i, \underline{\partial}_j \rangle f &= 0, & 1 \leq i < j \leq N. \end{aligned}$$

The vector space of simplicial harmonic polynomials, λ_i -homogeneous in the variable $\underline{u}_i$, will be denoted by $\mathcal{H}_{\lambda_1, \dots, \lambda_N}$ (with $\lambda_1 \geq \dots \geq \lambda_N \geq 0$).

Definition 2. A function $f : \mathbb{R}^{Nm} \rightarrow \mathbb{S}$, $(\underline{u}_1, \dots, \underline{u}_N) \mapsto f(\underline{u}_1, \dots, \underline{u}_N)$ is *simplicial monogenic* if the following conditions are satisfied:

$$\begin{aligned} \underline{\partial}_i f &= 0, & i &= 1, \dots, N \\ \langle \underline{u}_i, \underline{\partial}_j \rangle f &= 0, & 1 \leq i < j \leq N. \end{aligned}$$

The vector space of simplicial monogenic polynomials, λ_i -homogeneous in the variable $\underline{u}_i$, will be denoted $\mathcal{S}_{\lambda_1, \dots, \lambda_N}$ (with $\lambda_1 \geq \dots \geq \lambda_N \geq 0$).

In particular it was shown in [12] that the vector space $\mathcal{H}_{\lambda_1, \dots, \lambda_N}$ corresponds to the irreducible $\text{Spin}(m)$ -module with highest weight $(\lambda_1, \dots, \lambda_N)$, with respect to the regular representation on scalar-valued polynomials. The vector space $\mathcal{S}_{\lambda_1, \dots, \lambda_N}$ can be represented by the irreducible $\text{Spin}(m)$ -module with highest weight $(\lambda_1, \dots, \lambda_N)'$, with respect to the regular representation on spinor-valued polynomials. Taking $N = 1$ in these definitions, we reobtain the spherical harmonics $\mathcal{H}_{\lambda_1}$ of degree $\lambda_1 \in \mathbb{N}$ and the spherical monogenics $\mathcal{M}_{\lambda_1}$ of degree λ_1 , respectively.

From now on we take $N = 2$ and we put $(u_1, u_2) = (\underline{u}, \underline{v})$ in Definitions 1 and 2. In [2] we proved the generalization of the classical refined Fischer decomposition to the case of two Clifford variables:

Proposition 1. *For all integers $k \geq l \geq 0$ with $k > 0$, one has*

$$\mathcal{H}_{k,l} \otimes \mathbb{S} = \mathcal{S}_{k,l} \oplus (1 - \delta_{l,0})\mu_{k,l}\mathcal{S}_{k-1,l} \oplus (1 - \delta_{k,l})\nu_{k,l}\mathcal{S}_{k,l-1} \oplus (1 - \delta_{l,0})\kappa_{k,l}\mathcal{S}_{k-1,l-1}$$

with embedding factors

$$\begin{aligned}\mu_{k,l} &:= \underline{u} \\ \nu_{k,l} &:= \underline{v} - \frac{\underline{u}\langle \underline{v}, \underline{\partial}_u \rangle}{k - l + 1} \\ \kappa_{k,l} &:= \langle \underline{v}, \underline{u} \rangle + \frac{m + k + l - 4}{m + 2k - 4} \underline{v} \underline{u} - \frac{\langle \underline{u}, \underline{u} \rangle \langle \underline{v}, \underline{\partial}_u \rangle}{m + 2k - 4}.\end{aligned}$$

This proposition was used to define the unique (up to a multiplicative constant) elliptic, $\text{Spin}(m)$ -invariant, first-order differential operator $\mathcal{Q}_{k,l}$, acting between functions taking values in $\mathcal{S}_{k,l}$, as follows.

Definition 3. *For all integers $k \geq l \geq 0$ with $k > 0$, there exists a unique (up to a multiplicative constant) invariant first-order differential operator $\mathcal{Q}_{k,l}$ defined by*

$$\mathcal{Q}_{k,l} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) : f \mapsto \pi_{k,l}(\underline{\partial}_x f),$$

with $\pi_{k,l} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{H}_{k,l} \otimes \mathbb{S}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$ the projection operator defined by

$$\pi_{k,l} = \mathbf{1} + \frac{\underline{u} \underline{\partial}_u}{m + 2k - 2} + \frac{\underline{v} \underline{\partial}_v}{m + 2l - 4} - 2 \frac{\underline{u} \langle \underline{v}, \underline{\partial}_u \rangle \underline{\partial}_v}{(m + 2k - 2)(m + 2l - 4)}.$$

Remark 1. *In case $k = l > 0$, the operator reduces to*

$$\mathcal{Q}_{k,k} f = \left[\mathbf{1} + \frac{(\underline{v} - \underline{u} \langle \underline{v}, \underline{\partial}_u \rangle) \underline{\partial}_v}{m + 2k - 4} \right] \underline{\partial}_x f. \quad (1)$$

In case $l = 0$, we reobtain the Rarita-Schwinger operator $\mathcal{R}_k$ from [9]:

$$\mathcal{Q}_{k,0} = \mathcal{R}_k = \left(\mathbf{1} + \frac{\underline{u} \underline{\partial}_u}{m + 2k - 2} \right) \underline{\partial}_x. \quad (2)$$

Definition 4. *For all integers $k > l \geq 0$, the dual twistor operator $\mathbb{T}_{k-1,l}^{k,l}$ is defined as the unique (up to a multiplicative constant) invariant first-order differential operator*

$$\mathbb{T}_{k-1,l}^{k,l} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k-1,l}) : f \mapsto \left(\langle \underline{\partial}_u, \underline{\partial}_x \rangle + \frac{\langle \underline{v}, \underline{\partial}_u \rangle \langle \underline{\partial}_v, \underline{\partial}_x \rangle}{k - l + 1} \right) f.$$

In case $k = l > 0$, this operator does not exist.

Definition 5. *For all integers $k \geq l > 0$, the dual twistor operator $\mathbb{T}_{k,l-1}^{k,l}$ is defined as the unique (up to a multiplicative constant) invariant first-order differential operator*

$$\mathbb{T}_{k,l-1}^{k,l} : \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l-1}) : f \mapsto \langle \underline{\partial}_v, \underline{\partial}_x \rangle f.$$

In [2] we defined two types of polynomial null solutions of $\mathcal{Q}_{k,l}$, which are h -homogeneous in the vector variable $\underline{x}$. Only the case $h \geq k$ is considered in this paper. We investigate the (non-trivial) case $h < k$ in [4]. One type of null solutions is constructed through an inductive procedure, using information about the structure of $\text{Ker}_{h-1} \mathcal{Q}_{k-1,l}$, $\text{Ker}_{h-1} \mathcal{Q}_{k,l-1}$ and $\text{Ker}_{h-2} \mathcal{Q}_{k-1,l-1}$. The next proposition, proved in [2], will be very useful in this respect.

Proposition 2. *Let $f \in \text{Ker}_h \mathcal{Q}_{k,l}$.*

(i) *If $k \geq l > 0$, then $\mathbb{T}_{k,l-1}^{k,l} f \in \text{Ker}_{h-1} \mathcal{Q}_{k,l-1}$.*

(ii) *If $k > l \geq 0$, then $\mathbb{T}_{k-1,l}^{k,l} f \in \text{Ker}_{h-1} \mathcal{Q}_{k-1,l}$.*

Remark 2. *In this paper, the embedding factors of the $\text{Spin}(m)$ -irreducibles in the decomposition of $\text{Ker}_h \mathcal{Q}_{k,l}$ are not taken into account and the irreducible modules are systematically denoted by their highest weights only. These embedding factors are constructed in [3]. It follows from Proposition 2 that*

$$\text{Ker}_h \mathcal{Q}_{k,l} \cong \text{Ker} \mathbb{T}_{k-1,l}^{k,l} \oplus \text{Im} \mathbb{T}_{k-1,l}^{k,l} \cong \text{Ker} \mathbb{T}_{k,l-1}^{k,l} \oplus \text{Im} \mathbb{T}_{k,l-1}^{k,l} \quad (3)$$

where $\text{Im} \mathbb{T}_{k-1,l}^{k,l}$ and $\text{Im} \mathbb{T}_{k,l-1}^{k,l}$ should be thought of as isomorphic copies inside $\text{Ker}_h \mathcal{Q}_{k,l}$.

Our study of h -homogeneous polynomial null solutions of $\mathcal{Q}_{k,l}$ runs as follows. In Section 3 we discuss some useful dimensional results. An overview of the two types of null solutions of $\mathcal{Q}_{k,l}$ is given in Section 4 and a theorem about the structure of the kernel space is formulated. In Sections 5–9 we prove the decomposition of $\text{Ker}_h \mathcal{Q}_{k,l}$ into $\text{Spin}(m)$ -irreducible modules. This proof goes by induction on k and l . Finally, Section 10 deals with some examples.

3 The generalization of the classical CK-extension

For calculating the dimension of the vector space $\text{Ker}_h \mathcal{Q}_{k,l}$, we will use a generalized version of the classical CK-extension principle in Clifford analysis mentioned in Section 2. Let $f \in \text{Ker}_h \mathcal{Q}_{k,l}$ and consider the splitting

$$\underline{x} = \underline{x}^* + \underline{e}_m x_m \in \mathbb{R}^m = \mathbb{R}^{m-1} \oplus \mathbb{R}.$$

Define $\mathcal{Q}_{k,l}^* = \pi_{k,l}(\sum_{i=1}^{m-1} \underline{e}_i \partial_{x_i})$ as the higher spin operator on $\mathbb{R}^{m-1}$, i.e. the projection of the (classical) Dirac operator $\underline{\partial}_x^* = \sum_{i=1}^{m-1} \underline{e}_i \partial_{x_i}$ on $\mathbb{R}^{m-1}$. Then,

$$f \in \text{Ker}_h \mathcal{Q}_{k,l} \Leftrightarrow \partial_{x_m} f = -(\pi_{k,l}(\underline{e}_m))^{-1} \mathcal{Q}_{k,l}^* f.$$

If f^* denotes the restriction of f to the hyperplane $x_m = 0$, i.e. $f(\underline{x}^*, 0) = f^*(\underline{x}^*)$, the unique solution of the first order differential equation above satisfying this initial value condition, is given by

$$f(\underline{x}; \underline{u}, \underline{v}) = \exp \left(-x_m (\pi_{k,l}(\underline{e}_m))^{-1} \mathcal{Q}_{k,l}^* \right) f^*(\underline{x}^*; \underline{u}, \underline{v})$$

and is referred to as the generalized CK-extension of f^* . It defines a null solution f for the operator $\mathcal{Q}_{k,l}$ in an open neighbourhood in $\mathbb{R}^m$ of $\mathbb{R}^{m-1}$, such that its restriction to the hyperplane $x_m = 0$ equals f^* . This CK-extension defines an isomorphism between the space of h -homogeneous polynomials on $\mathbb{R}^{m-1}$ taking values in $\mathcal{S}_{k,l}$, and the space of h -homogeneous null solutions of the operator $\mathcal{Q}_{k,l}$ on $\mathbb{R}^m$. In particular,

$$\dim_{\mathbb{C}} \text{Ker}_h \mathcal{Q}_{k,l} = \dim_{\mathbb{C}} \mathcal{P}_h(\mathbb{R}^{m-1}, \mathcal{S}_{k,l}) = \dim_{\mathbb{C}} \mathcal{P}_h(\mathbb{R}^{m-1}) \dim_{\mathbb{C}} \mathcal{S}_{k,l}. \quad (4)$$

The dimension of the irreducible $\text{Spin}(m)$ -modules $\mathcal{S}_{k,l}$ can be calculated using the Weyl dimension formula (see e.g. [16]) resulting into

$$\dim_{\mathbb{C}} \mathcal{S}_{k,l} = 2^n \binom{k+2n-2}{k+1} \binom{l+2n-3}{l} \frac{(2n+k+l-1)(k-l+1)}{(2n-1)(2n-2)}. \quad (5)$$

In what follows, we will also need

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{S}_{h,k,l} &= 2^n \binom{h+2n-3}{h+2} \binom{k+2n-4}{k+1} \binom{l+2n-5}{l} \\ &\quad \cdot (2n+h+k-1)(2n+h+l-2)(2n+k+l-3) \\ &\quad \cdot (h-k+1)(h-l+2)(k-l+1) \frac{(2n-5)! (2n-5)!}{(2n-1)! (2n-3)!}. \end{aligned} \quad (6)$$

4 Two types of null solutions for $\mathcal{Q}_{k,l}$

A function f will be in the kernel space of the operator $\mathcal{Q}_{k,l}$, whenever

$$\mathcal{Q}_{k,l} f = 0 \Leftrightarrow \pi_{k,l}(\underline{\partial}_x f) = 0.$$

As there are two possibilities to satisfy this condition, there are two types of homogeneous polynomial null solutions f for $\mathcal{Q}_{k,l}$: either $\underline{\partial}_x f = 0$ (called type A solutions) or $\underline{\partial}_x f \neq 0$ but $\pi_{k,l}(\underline{\partial}_x f) = 0$ (called type B solutions). A detailed description of both types of solutions can be found in [2]; let us mention here the main results.

4.1 Solutions of Type A

For all integers $h \geq k \geq l > 0$, define $\mathcal{P}_{h,k,l}(\mathbb{S})$ to be the space of $\mathbb{S}$ -valued polynomials in three vector variables $(\underline{x}, \underline{u}, \underline{v})$, homogeneous of degree h , k and l in $\underline{x}$, $\underline{u}$ and $\underline{v}$ respectively. Denote the space of so-called triple monogenics by

$$\mathcal{M}_{h,k,l} = \{f \in \mathcal{P}_{h,k,l}(\mathbb{S}) \mid \underline{\partial}_x f = \underline{\partial}_u f = \underline{\partial}_v f = 0\}.$$

This vector space is reducible w.r.t. the action of $\text{Spin}(m)$, and in [5] the decomposition of this space in terms of irreducible $\text{Spin}(m)$ -modules was determined. Define $\mathcal{M}_{h,k,l}^s$ to be the space $\mathcal{M}_{h,k,l} \cap \text{Ker} \langle \underline{u}, \underline{\partial}_v \rangle$, i.e. $\mathcal{M}_{h,k,l}^s = \{f \in \mathcal{M}_{h,k,l} \mid \langle \underline{u}, \underline{\partial}_v \rangle f = 0\}$. This intersection ensures that $\mathcal{M}_{h,k,l}^s \subset \mathcal{C}^\infty(\mathbb{R}^m; \mathcal{S}_{k,l})$. By construction, $\mathcal{M}_{h,k,l}^s$ is the space of

h -homogeneous type A solutions for $\mathcal{Q}_{k,l}$. Also in [5] we proved that this space decomposes into $\text{Spin}(m)$ -irreducibles as

$$\mathcal{M}_{h,k,l}^s = \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^l (h+i+j, k-i, l-j)'. \quad (7)$$

In the special case $l = 0$, we write

$$\mathcal{M}_{h,k} := \mathcal{M}_{h,k,0}^s = \bigoplus_{i=0}^k (h+i, k-i)'. \quad (8)$$

Remark 3. Every irreducible module in $\mathcal{M}_{h,k,l}^s$ and $\mathcal{M}_{h,k,k}^s$ appears with multiplicity one. A necessary condition for a module $\mathcal{S}_{p,q,r}$ to be in the decomposition of $\mathcal{M}_{h,k,l}^s$ (up to an isomorphic copy) is $p+q+r = h+k+l$.

4.2 Solutions of Type B

A different approach, based on an inductive procedure, is needed to describe the type B solutions. The definition of $\mathcal{Q}_{k,l}$ leads to

$$\begin{aligned} \mathcal{Q}_{k,l}f = 0 &\Leftrightarrow \left[1 + \frac{u \partial_u}{m+2k-2} + \frac{v \partial_v}{m+2l-4} - 2 \frac{u \langle v, \partial_u \rangle \partial_v}{(m+2k-2)(m+2l-4)} \right] \partial_x f = 0 \\ &\Leftrightarrow \partial_x f = \mu_{k,l} \frac{2}{m+2k-2} \mathbb{T}_{k-1,l}^{k,l} f + \nu_{k,l} \frac{2}{m+2l-4} \mathbb{T}_{k,l-1}^{k,l} f. \end{aligned} \quad (9)$$

In view of Proposition 2, the following implication thus holds:

$$\mathcal{Q}_{k,l}f = 0 \Rightarrow \partial_x f = \mu_{k,l} g_1 + \nu_{k,l} g_2, \quad (10)$$

with $g_1 \in \text{Ker}_{h-1} \mathcal{Q}_{k-1,l}$ and $g_2 \in \text{Ker}_{h-1} \mathcal{Q}_{k,l-1}$. Conversely, let $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathcal{S}_{k,l})$ be a polynomial, h -homogeneous in $\underline{x}$, such that g_1 and g_2 satisfy the requirements mentioned above and with

$$\partial_x f = \mu_{k,l} g_1 + \nu_{k,l} g_2, \quad (11)$$

then $f \in \text{Ker}_h \mathcal{Q}_{k,l}$. Now we would like to investigate whether for any choice of these polynomials g_1 and g_2 , there indeed *exists* a polynomial f satisfying (11). In other words, we are trying to characterize the conditions which have to be imposed on $g_1 \in \text{Ker}_{h-1} \mathcal{Q}_{k-1,l}$ and $g_2 \in \text{Ker}_{h-1} \mathcal{Q}_{k,l-1}$, such that the following equivalence holds:

$$f \in \text{Ker}_h \mathcal{Q}_{k,l} \Leftrightarrow \begin{cases} \partial_x f = \mu_{k,l} g_1 + \nu_{k,l} g_2 \\ \partial_u f = 0 \\ \partial_v f = 0 \\ \langle u, \partial_v \rangle f = 0 \end{cases}.$$

This requires the study of compatibility conditions for an inhomogeneous system of equations involving three Dirac operators, see [11] for details. In [2] we proved that for $g_1 \in \text{Ker}_{h-1} \mathcal{Q}_{k-1,l}$ and $g_2 \in \text{Ker}_{h-1} \mathcal{Q}_{k,l-1}$ satisfying

$$\mathbb{T}_{k-1,l-1}^{k-1,l} g_1 = \frac{(m+2l-4)(k-l+2)}{(m+2k-2)(k-l+1)} \mathbb{T}_{k-1,l-1}^{k,l-1} g_2 \quad (12)$$

there exists a polynomial f in $\text{Ker}_h \mathcal{Q}_{k,l}$ such that $\partial_x f = \mu_{k,l} g_1 + \nu_{k,l} g_2$. This means that g_1 and g_2 satisfying this relation can be *inverted* in order to obtain an element in $\text{Ker}_h \mathcal{Q}_{k,l}$, for details we refer to [2]. Note that both the left- and right-hand side of (12) are polynomials in $\text{Ker}_{h-2} \mathcal{Q}_{k-1,l-1}$.

The type B null solutions of the operator $\mathcal{Q}_{k,l}$ can be listed as follows:

- (i) choosing $g_2 = 0$, we have that $\text{Ker}_{h-1} \mathcal{Q}_{k-1,l} \cap \text{Ker} \mathbb{T}_{k-1,l-1}^{k-1,l} \subset \text{Ker}_h \mathcal{Q}_{k,l}$;
- (ii) choosing $g_1 = 0$, we have that $\text{Ker}_{h-1} \mathcal{Q}_{k,l-1} \cap \text{Ker} \mathbb{T}_{k-1,l-1}^{k,l-1} \subset \text{Ker}_h \mathcal{Q}_{k,l}$;
- (iii) finally, choosing both g_1 and $g_2 \neq 0$ is only possible if relation (12) is satisfied, which amounts to saying that certain elements in $\text{Ker}_{h-2} \mathcal{Q}_{k-1,l-1}$ can *also* be inverted. These summands could be described as $\text{Im} \mathbb{T}_{k-1,l-1}^{k,l-1} \cap \text{Im} \mathbb{T}_{k-1,l-1}^{k-1,l} \subset \text{Ker}_{h-2} \mathcal{Q}_{k-1,l-1}$.

In the special case $k = l$, there exists only one twistor operator and (12) reduces to $\mathbb{T}_{k-1,k-1}^{k,k-1} g_2 = 0$. Hence, the type B solutions of the operator $\mathcal{Q}_{k,k}$ are elements of $\text{Ker}_{h-1} \mathcal{Q}_{k,k-1} \cap \text{Ker} \mathbb{T}_{k-1,k-1}^{k,k-1}$.

We thus have obtained an inductive procedure to describe the space of polynomial null solutions for the operator $\mathcal{Q}_{k,l}$. Together with the type A solutions, the kernel space of h -homogeneous polynomial null solutions for $\mathcal{Q}_{k,l}$ can be constructed as follows.

Theorem 1. *For $h > k \geq l$, the kernel space $\text{Ker}_h \mathcal{Q}_{k,l}$ for the invariant first-order operator $\mathcal{Q}_{k,l}$ decomposes as*

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{k,l} = & \mathcal{M}_{h,k,l}^s \oplus \left[\text{Ker}_{h-1} \mathcal{Q}_{k-1,l} \cap \text{Ker} \mathbb{T}_{k-1,l-1}^{k-1,l} \right] \\ & \oplus \left[\text{Ker}_{h-1} \mathcal{Q}_{k,l-1} \cap \text{Ker} \mathbb{T}_{k-1,l-1}^{k,l-1} \right] \oplus \left[\text{Im} \mathbb{T}_{k-1,l-1}^{k,l-1} \cap \text{Im} \mathbb{T}_{k-1,l-1}^{k-1,l} \right]. \end{aligned} \quad (13)$$

The operators and kernel spaces in Theorem 1 are visualized in the following diagram:

$$\begin{array}{ccc} \text{Ker}_{h-1} \mathcal{Q}_{k,l-1} & \xleftarrow{\mathbb{T}_{k,l-1}^{k,l}} & \text{Ker}_h \mathcal{Q}_{k,l} \\ \mathbb{T}_{k-1,l-1}^{k,l-1} \downarrow & & \downarrow \mathbb{T}_{k-1,l}^{k,l} \\ \text{Ker}_{h-2} \mathcal{Q}_{k-1,l-1} & \xleftarrow{\mathbb{T}_{k-1,l-1}^{k-1,l}} & \text{Ker}_{h-1} \mathcal{Q}_{k-1,l} \end{array} \quad .$$

Our results are confirmed by our experiments with LiE, the computer algebra package for Lie group computations [19]. To construct $\text{Ker}_h \mathcal{Q}_{k,l}$, we used the following result from [22], translated to the present case:

$$\text{Ker}_h \mathcal{Q}_{k,l} = \mathcal{P}_h \otimes \mathcal{S}_{k,l} \bmod \mathcal{P}_{h-1} \otimes \mathcal{S}_{k,l},$$

where $\mathcal{P}_h$ denotes the space of h -homogeneous polynomials, and we used LiE to calculate these tensor products.

At this point, it should be emphasized that we still have to show that there is indeed equality in the decomposition (13). We will prove this by a dimensional analysis. The dimension of the left-hand side being known by means of the generalized CK-extension, the real problem is counting the dimension of the right-hand side of (13). To that end we decompose this right-hand side into $\text{Spin}(m)$ -irreducibles. This is done in several steps, proceeding by induction on k and l .

- (i) The case $l = 0$, i.e. the case of the Rarita-Schwinger operator $\mathcal{R}_k$, was studied in [9, 10] and is briefly recalled in Section 5.
- (ii) Section 6 deals with the description of $\text{Ker}_h \mathcal{Q}_{1,1}$, $\text{Ker}_h \mathcal{Q}_{2,1}$ and $\text{Ker}_h \mathcal{Q}_{3,1}$. This allows for inspiring an induction hypothesis to prove the general case $\text{Ker}_h \mathcal{Q}_{k,1}$ by induction on k .
- (iii) We use the results of the previous sections to formulate an induction hypothesis for the structure of $\text{Ker}_h \mathcal{Q}_{k,l-1}$, $k \geq l-1$ in Section 7. This leads to the construction of $\text{Ker}_h \mathcal{Q}_{l,l}$ in Section 8.
- (iv) Finally, in Section 9, we decompose $\text{Ker}_h \mathcal{Q}_{k,l}$ into irreducible $\text{Spin}(m)$ -modules. The proof goes by induction on k , after having proved the decomposition of $\text{Ker}_h \mathcal{Q}_{l+1,l}$ as the basic step.

5 Decomposition of $\text{Ker}_h \mathcal{R}_k$

It was shown in [9] how $\text{Ker}_h \mathcal{R}_k$ decomposes into irreducible $\text{Spin}(m)$ -modules, denoted by their highest weight for convenience:

$$\text{Ker}_h \mathcal{R}_k = \bigoplus_{i=0}^k \mathcal{M}_{h-i, k-i} = \bigoplus_{i=0}^k \bigoplus_{j=0}^{k-i-j} (h-i+j, k-i-j)', \quad (14)$$

The space $\mathcal{M}_{h,k}$ coincides with the h -homogeneous type A solutions for $\mathcal{R}_k$. Recall that $\mathcal{M}_{h,k}$ contains solutions to both $\mathcal{R}_k$ and the operator $\mathbb{T}_{k-1,0}^{k,0} \equiv \langle \partial_u, \partial_x \rangle$. Hence it follows from (3) that

Lemma 1. *For all integers $h \geq k$ one has*

$$\begin{aligned} \text{Ker}_h \mathcal{R}_k \cap \text{Ker } \mathbb{T}_{k-1,0}^{k,0} &= \mathcal{M}_{h,k} \\ \text{Im } \mathbb{T}_{k-1,0}^{k,0} &= \bigoplus_{i=1}^k \mathcal{M}_{h-i,k-i} = \bigoplus_{i=0}^{k-1} \mathcal{M}_{h-1-i,k-1-i}. \end{aligned}$$

6 Decomposition of $\text{Ker}_h \mathcal{Q}_{k,1}$

6.1 Decomposition of $\text{Ker}_h \mathcal{Q}_{1,1}$

We begin the proof of the decomposition of $\text{Ker}_h \mathcal{Q}_{k,1}$ by taking $k = 1$. This is a special case of $\text{Ker}_h \mathcal{Q}_{l,l}$, which is discussed in Section 8. Using (7) and (8), we write the result from [8] as

$$\text{Ker}_h \mathcal{Q}_{1,1} = \mathcal{M}_{h,1,1}^s \oplus \mathcal{M}_{h-1,1} = \bigoplus_{i=0}^1 \bigoplus_{j=0}^1 (h+i-j, 1-i, 1-j)'. \quad (15)$$

There holds

Lemma 2. *For all integers $h \geq 1$ one has*

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{1,1} \cap \text{Ker } \mathbb{T}_{1,0}^{1,1} &= \mathcal{M}_{h,1,1}^s \\ \text{Im } \mathbb{T}_{1,0}^{1,1} &= \mathcal{M}_{h-1,1}. \end{aligned}$$

Proof. We have from (9) that

$$\mathcal{Q}_{1,1} f = 0 \Leftrightarrow \underline{\partial}_x f = \frac{2}{m-2} \nu_{1,1} \mathbb{T}_{1,0}^{1,1} f,$$

proving the first results. The second result then follows from (3). $\square$

6.2 Decomposition of $\text{Ker}_h \mathcal{Q}_{2,1}$

By means of Lemma 1, Lemma 2 and Theorem 1 we can prove the following

Proposition 3. *For $h > 2$, the space $\text{Ker}_h \mathcal{Q}_{2,1}$ decomposes as*

$$\text{Ker}_h \mathcal{Q}_{2,1} = \mathcal{M}_{h,2,1}^s \oplus \mathcal{M}_{h-1,1,1}^s \oplus \mathcal{M}_{h-1,2} \oplus \mathcal{M}_{h-2,1}.$$

Proof. A dimensional analysis shows that there are no more other irreducibles in $\text{Ker}_h \mathcal{Q}_{2,1}$ than the ones in the right-hand side of this decomposition. For the dimension of the left-hand side, the generalization of the CK-extension leads to

$$\dim_{\mathbb{C}} \text{Ker}_h \mathcal{Q}_{2,1} = 2^n \binom{h+2n-1}{h} \frac{(2n+2)2n(2n-2)}{3}.$$

The dimension of the right-hand side is calculated as the sum of the dimensions of the irreducible $\text{Spin}(m)$ -modules:

$$\begin{aligned} & \dim_{\mathbb{C}} \mathcal{S}_{h,2,1} + \dim_{\mathbb{C}} \mathcal{S}_{h+1,1,1} + \dim_{\mathbb{C}} \mathcal{S}_{h+1,2} + \dim_{\mathbb{C}} \mathcal{S}_{h+2,1} + \dim_{\mathbb{C}} \mathcal{S}_{h-1,1,1} \\ & + \dim_{\mathbb{C}} \mathcal{S}_{h,1} + \dim_{\mathbb{C}} \mathcal{S}_{h-1,2} + \dim_{\mathbb{C}} \mathcal{S}_{h,1} + \dim_{\mathbb{C}} \mathcal{M}_{h+1} + \dim_{\mathbb{C}} \mathcal{S}_{h-2,1} + \dim_{\mathbb{C}} \mathcal{M}_{h-1}. \end{aligned}$$

Using Maple, see [20], together with the formulas (5) and (6), it is shown that this sum indeed equals $\dim_{\mathbb{S}} \text{Ker}_h \mathcal{Q}_{2,1}$. $\square$

Now we will prove that

Lemma 3. *For every integer $h \geq 2$, one has*

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{2,1} \cap \text{Ker } \mathbb{T}_{2,0}^{2,1} &= \mathcal{M}_{h,2,1}^s \oplus \mathcal{M}_{h-1,1,1}^s \\ \text{Im } \mathbb{T}_{2,0}^{2,1} &= \mathcal{M}_{h-1,2} \oplus \mathcal{M}_{h-2,1}. \end{aligned}$$

Proof. Recall from (9) that

$$\mathcal{Q}_{2,1} f = 0 \Leftrightarrow \underline{\partial}_x f = \mu_{2,1} \frac{2}{m+2} \mathbb{T}_{1,1}^{2,1} f + \nu_{2,1} \frac{2}{m-2} \mathbb{T}_{2,0}^{2,1} f.$$

For $f \in \text{Ker}_h \mathcal{Q}_{2,1} \cap \text{Ker } \mathbb{T}_{2,0}^{2,1}$, this leads to

$$\underline{\partial}_x f = \mu_{2,1} \frac{2}{m+2} \mathbb{T}_{1,1}^{2,1} f = \frac{2}{m+2} u \langle \underline{\partial}_u, \underline{\partial}_x \rangle f,$$

which is definitely fulfilled if $f \in \mathcal{M}_{h,2,1}^s$. If $\underline{\partial}_x f \neq 0$, then $\mathbb{T}_{1,1}^{2,1} f \in \text{Ker}_{h-1} \mathcal{Q}_{1,1}$ by Proposition 2 and

$$\mathcal{Q}_{1,1} \mathbb{T}_{1,1}^{2,1} f = 0 \Leftrightarrow \underline{\partial}_x \langle \underline{\partial}_u, \underline{\partial}_x \rangle f = 0 \Leftrightarrow \langle \underline{\partial}_u, \underline{\partial}_x \rangle f \in \mathcal{M}_{h-1,1,1}^s.$$

The statement in the lemma then follows then Proposition 3 and (3). $\square$

In view of formulating a hypothesis for the induction principle, the result of the previous proposition is rewritten as

$$\text{Ker}_h \mathcal{Q}_{2,1} = \bigoplus_{i=0}^1 \bigoplus_{j=0}^1 \mathcal{M}_{h-i-j, 2-i, 1-j}^s. \quad (16)$$

6.3 Decomposition of $\text{Ker}_h \mathcal{Q}_{3,1}$

Taking $k = 3$ and $l = 1$, Theorem 1 leads to

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{3,1} &= \mathcal{M}_{h,3,1}^s \oplus \left[\text{Ker}_{h-1} \mathcal{Q}_{2,1} \cap \text{Ker } \mathbb{T}_{2,0}^{2,1} \right] \oplus \left[\text{Ker}_{h-1} \mathcal{R}_3 \cap \text{Ker } \mathbb{T}_{2,0}^{3,0} \right] \\ &\oplus \left[\text{Im } \mathbb{T}_{2,0}^{3,0} \cap \text{Im } \mathbb{T}_{2,0}^{2,1} \right]. \end{aligned}$$

It thus follows from Lemma 1 and Lemma 3 that

Proposition 4. For $h > 3$, the decomposition of the space $\text{Ker}_h \mathcal{Q}_{3,1}$ is given by

$$\text{Ker}_h \mathcal{Q}_{3,1} = \mathcal{M}_{h,3,1}^s \oplus \mathcal{M}_{h-1,3} \oplus \bigoplus_{i=0}^1 \mathcal{M}_{h-1-i,2-i,1}^s \oplus \bigoplus_{i=1}^2 \mathcal{M}_{h-1-i,3-i}.$$

Proof. As in the previous proposition, the proof goes by counting dimensions of the left-hand side and the right-hand side independently. $\square$

Again with an eye to formulating an induction hypothesis, this result may be rewritten as

$$\text{Ker}_h \mathcal{Q}_{3,1} = \bigoplus_{i=0}^2 \bigoplus_{j=0}^1 \mathcal{M}_{h-i-j,3-i,1-j}^s. \quad (17)$$

6.4 Decomposition of $\text{Ker}_h \mathcal{Q}_{k,1}$

The results in (15), (16) and (17) inspire the following induction hypothesis:

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{k-1,1} &= \bigoplus_{i=0}^{k-2} \bigoplus_{j=0}^1 \mathcal{M}_{h-i-j,k-1-i,1-j}^s \\ &= \bigoplus_{i=0}^{k-2} \mathcal{M}_{h-i,k-1-i,1}^s \oplus \bigoplus_{i=0}^{k-2} \mathcal{M}_{h-1-i,k-1-i}. \end{aligned} \quad (18)$$

Assuming this hypothesis to hold, we can prove the following lemma.

Lemma 4. For every integer $h \geq k-1$ one has

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{k-1,1} \cap \text{Ker} \mathbb{T}_{k-1,0}^{k-1,1} &= \bigoplus_{i=0}^{k-2} \mathcal{M}_{h-i,k-1-i,1}^s \\ \text{Im} \mathbb{T}_{k-1,0}^{k-1,1} &= \bigoplus_{i=0}^{k-2} \mathcal{M}_{h-1-i,k-1-i}. \end{aligned}$$

Proof. The proof follows from the induction hypothesis made above, the result (3) and the fact that $\text{Im} \mathbb{T}_{k-1,0}^{k-1,1} \subset \text{Ker}_{h-1} \mathcal{R}_{k-1}$ where

$$\text{Ker}_{h-1} \mathcal{R}_{k-1} = \bigoplus_{i=0}^{k-1} \mathcal{M}_{h-1-i,k-1-i},$$

in view of Lemma 1. $\square$

Proposition 5. For $h > k$, the kernel space $\text{Ker}_h \mathcal{Q}_{k,1}$ decomposes as

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{k,1} &= \bigoplus_{i=0}^{k-1} \bigoplus_{j=0}^1 \mathcal{M}_{h-i-j,k-i,1-j}^s \\ &= \bigoplus_{i=0}^{k-1} \bigoplus_{j=0}^1 \bigoplus_{p=0}^{k-1-i+j} \bigoplus_{q=0}^{1-j} (h-i-j+p+q, k-i-p, 1-j-q)'. \end{aligned}$$

Proof. We will prove this by induction on k . Suppose that the induction hypothesis (19) holds. Then Lemma 4 holds, and its result is rewritten as

$$\begin{aligned} \text{Ker}_{h-1} \mathcal{Q}_{k-1,1} \cap \text{Ker} \mathbb{T}_{k-1,0}^{k-1,1} &= \bigoplus_{i=1}^{k-1} \mathcal{M}_{h-i,k-i,1}^s \\ \text{Im} \mathbb{T}_{k-1,0}^{k-1,1} &= \bigoplus_{i=1}^{k-1} \mathcal{M}_{h-1-i,k-i}. \end{aligned}$$

Using Lemma 1, the right-hand side of the formula in Theorem 1 leads to

$$\begin{aligned} &\mathcal{M}_{h,k,1}^s \oplus \left[\text{Ker}_{h-1} \mathcal{Q}_{k-1,1} \cap \text{Ker} \mathbb{T}_{k-1,0}^{k-1,1} \right] \oplus \left[\text{Ker}_{h-1} \mathcal{R}_k \cap \text{Ker} \mathbb{T}_{k-1,0}^{k,0} \right] \\ &\oplus \left[\text{Im} \mathbb{T}_{k-1,0}^{k,0} \cap \text{Im} \mathbb{T}_{k-1,0}^{k-1,0} \right] \\ &= \mathcal{M}_{h,k,1}^s \oplus \bigoplus_{i=1}^{k-1} \mathcal{M}_{h-i,k-i,1}^s \oplus \mathcal{M}_{h-1,k} \oplus \bigoplus_{i=1}^{k-1} \mathcal{M}_{h-1-i,k-i} \\ &= \bigoplus_{i=0}^{k-1} \bigoplus_{j=0}^1 \mathcal{M}_{h-i-j,k-i,1-j}^s. \end{aligned}$$

To prove that

$$\text{Ker}_h \mathcal{Q}_{k,1} = \bigoplus_{i=0}^{k-1} \bigoplus_{j=0}^1 \mathcal{M}_{h-i-j,k-i,1-j}^s$$

we verified, using Maple and the results from Section 3, that the dimensions of both sides are equal. $\square$

7 Induction hypothesis

Now that we have described the kernel space of the operators $\mathcal{R}_k$ (for every $k \geq 0$) and $\mathcal{Q}_{k,1}$ (for every $k \geq 1$), we formulate the following induction hypothesis on $l-1$:

$$\text{Ker}_h \mathcal{Q}_{k,l-1} = \bigoplus_{i=0}^{k-l+1} \bigoplus_{j=0}^{l-1} \mathcal{M}_{h-i-j,k-i,l-1-j}^s \quad (19)$$

for every $k \geq l-1$. Assuming this hypothesis we can prove the following lemma.

Lemma 5. *For every integer $h \geq k > l-1$ we have*

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{k,l-1} \cap \text{Ker} \mathbb{T}_{k-1,l-1}^{k,l-1} &= \bigoplus_{j=0}^{l-1} \mathcal{M}_{h-j,k,l-1-j}^s \\ \text{Im} \mathbb{T}_{k-1,l-1}^{k,l-1} &= \bigoplus_{i=1}^{k-l+1} \bigoplus_{j=0}^{l-1} \mathcal{M}_{h-i-j,k-i,l-1-j}^s. \end{aligned}$$

Proof. Note that we can write (19) as

$$\text{Ker}_h \mathcal{Q}_{k,l-1} = \bigoplus_{j=0}^{l-1} \mathcal{M}_{h-j,k,l-1-j}^s \oplus \bigoplus_{i=1}^{k-l+1} \bigoplus_{j=0}^{l-1} \mathcal{M}_{h-i-j,k-i,l-1-j}^s$$

The result then follows from (3) and the fact that $\text{Im } \mathbb{T}_{k-1,l-1}^{k,l-1} \subset \text{Ker}_{h-1} \mathcal{Q}_{k-1,l-1}$ with

$$\text{Ker}_{h-1} \mathcal{Q}_{k-1,l-1} = \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^{l-1} \mathcal{M}_{h-1-i-j,k-1-i,l-1-j}^s = \bigoplus_{i=1}^{k-l+1} \bigoplus_{j=0}^{l-1} \mathcal{M}_{h-i-j,k-i,l-1-j}^s,$$

according to the induction hypothesis made. $\square$

Let us for a moment recall the whole of our induction argument, which can be visualized in the following diagram.

The column $l = 0$ represents the known Rarita-Schwinger case. In the column $l = 1$ we have established the explicit decomposition of $\text{Ker}_h \mathcal{Q}_{k,1}$ for $k = 1, 2, 3$ and proved it for all $k > 1$ by induction on k . At this point we assume the decomposition theorem valid for all columns up to and including $l - 1$ and all $k \geq l - 1$. Then we are able to prove it for the column l : for $k = l$ in Section 8, and finally for $k > l$ in Section 9.

8 Decomposition of $\text{Ker}_h \mathcal{Q}_{l,l}$

We use the induction hypothesis stated in the previous section to describe $\text{Ker}_h \mathcal{Q}_{l,l}$.

Proposition 6. *For $h \geq l$, one has*

$$\text{Ker}_h \mathcal{Q}_{l,l} = \bigoplus_{j=0}^l \mathcal{M}_{h-j,l,l-j}^s = \bigoplus_{j=0}^l \bigoplus_{p=0}^j \bigoplus_{q=0}^{l-j} (h-j+p+q, l-p, l-j-q)'$$

Proof. The induction hypothesis (19) leads, via Lemma 5, to

$$\text{Ker}_{h-1} \mathcal{Q}_{l,l-1} \cap \text{Ker} \mathbb{T}_{l-1,l-1}^{l,l-1} = \bigoplus_{j=1}^l \mathcal{M}_{h-j,l,l-j}^s.$$

By the right-hand side of the formula in Theorem 1 we obtain

$$\mathcal{M}_{h,l,l}^s \oplus \left[\text{Ker}_{h-1} \mathcal{Q}_{l,l-1} \cap \text{Ker} \mathbb{T}_{l-1,l-1}^{l,l-1} \right] = \bigoplus_{j=0}^l \mathcal{M}_{h-j,l,l-j}^s.$$

Finally, a dimensional analysis of the left- and right-hand side shows that

$$\dim_{\mathbb{C}} \text{Ker}_h \mathcal{Q}_{l,l} = \sum_{j=0}^l \dim_{\mathbb{C}} \mathcal{M}_{h-j,l,l-j}^s = \sum_{j=0}^l \sum_{p=0}^j \sum_{q=0}^{l-j} \dim_{\mathbb{C}} \mathcal{S}_{h-j+p+q,l-p,l-j-q},$$

proving the statement. $\square$

Note that every irreducible module appears just once in this decomposition.

The proof of the following lemma is similar the proof of Lemma 2.

Lemma 6. *For all integers $h \geq l$ we have*

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{l,l} \cap \text{Ker} \mathbb{T}_{l,l-1}^{l,l} &= \mathcal{M}_{h,l,l}^s \\ \text{Im} \mathbb{T}_{l,l-1}^{l,l} &= \bigoplus_{j=1}^l \mathcal{M}_{h-j,l,l-j}^s. \end{aligned}$$

9 Decomposition of $\text{Ker}_h \mathcal{Q}_{k,l}$

To construct $\text{Ker}_h \mathcal{Q}_{k,l}$, we use the result of the previous section for $\text{Ker}_h \mathcal{Q}_{l,l}$, and proceed by induction on k . That is, we first perform a basic step, taking $k = l + 1$, followed by an inductive step. So the first step is the construction of $\text{Ker}_h \mathcal{Q}_{l+1,l}$. Using Theorem 1, Lemma 5 and Lemma 6, we obtain

$$\begin{aligned} & \mathcal{M}_{h,l+1,l}^s \oplus \left[\text{Ker}_{h-1} \mathcal{Q}_{l,l} \cap \text{Ker} \mathbb{T}_{l,l-1}^{l,l} \right] \\ & \oplus \left[\text{Ker}_{h-1} \mathcal{Q}_{l+1,l-1} \cap \text{Ker} \mathbb{T}_{l,l-1}^{l+1,l-1} \right] \oplus \left[\text{Im} \mathbb{T}_{l,l-1}^{l+1,l-1} \cap \text{Im} \mathbb{T}_{l,l-1}^{l,l} \right] \\ &= \mathcal{M}_{h,l+1,l}^s \oplus \mathcal{M}_{h-1,l,l}^s \oplus \bigoplus_{j=1}^l \mathcal{M}_{h-j,l+1,l-j}^s \oplus \bigoplus_{j=1}^l \mathcal{M}_{h-j,l,l-j}^s \\ &= \bigoplus_{i=0}^1 \bigoplus_{j=0}^l \mathcal{M}_{h-i-j,l+1-i,l-j}^s. \end{aligned}$$

A dimensional analysis shows that indeed

$$\text{Ker}_h \mathcal{Q}_{l+1,l} = \bigoplus_{i=0}^1 \bigoplus_{j=0}^l \mathcal{M}_{h-i-j, l+1-i, l-j}^s.$$

This allows us to formulate an induction hypothesis on $k-1$:

$$\text{Ker}_h \mathcal{Q}_{k-1,l} = \bigoplus_{i=0}^{k-1-l} \bigoplus_{j=0}^l \mathcal{M}_{h-i-j, k-1-i, l-j}^s, \quad (20)$$

whence, assuming this hypothesis to hold,

Lemma 7. *For $h \geq k-1$ one has*

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{k-1,l} \cap \text{Ker} \mathbb{T}_{k-1, l-1}^{k-1,l} &= \bigoplus_{i=0}^{k-1-l} \mathcal{M}_{h-i, k-1-i, l}^s \\ \text{Im} \mathbb{T}_{k-1, l-1}^{k-1,l} &= \bigoplus_{i=0}^{k-1-l} \bigoplus_{j=1}^l \mathcal{M}_{h-i-j, k-1-i, l-j}^s. \end{aligned}$$

Proof. We write the induction hypothesis (20) as

$$\text{Ker}_h \mathcal{Q}_{k-1,l} = \bigoplus_{i=0}^{k-1-l} \mathcal{M}_{h-i, k-1-i, l}^s \oplus \bigoplus_{i=0}^{k-1-l} \bigoplus_{j=1}^l \mathcal{M}_{h-i-j, k-1-i, l-j}^s.$$

The proof then follows from (3) and the fact that $\text{Im} \mathbb{T}_{k-1, l-1}^{k-1,l} \subset \text{Ker}_{h-1} \mathcal{Q}_{k-1, l-1}$, by Proposition 2. Note that

$$\text{Ker}_{h-1} \mathcal{Q}_{k-1, l-1} = \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^{l-1} \mathcal{M}_{h-1-i-j, k-1-i, l-1-j}^s = \bigoplus_{i=0}^{k-l} \bigoplus_{j=1}^l \mathcal{M}_{h-i-j, k-1-i, l-j}^s,$$

according to the induction hypothesis of Section 7. □

Note that the results of this lemma may be rewritten as

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{k-1,l} \cap \text{Ker} \mathbb{T}_{k-1, l-1}^{k-1,l} &= \bigoplus_{i=1}^{k-l} \mathcal{M}_{h+1-i, k-i, l}^s \\ \text{Im} \mathbb{T}_{k-1, l-1}^{k-1,l} &= \bigoplus_{i=1}^{k-l} \bigoplus_{j=1}^l \mathcal{M}_{h+1-i-j, k-i, l-j}^s. \end{aligned}$$

Together with Lemma 5, we find

$$\text{Im} \mathbb{T}_{k-1, l-1}^{k, l-1} \cap \text{Im} \mathbb{T}_{k-1, l-1}^{k-1, l} = \bigoplus_{i=1}^{k-l} \bigoplus_{j=1}^l \mathcal{M}_{h-i-j, k-i, l-j}^s.$$

Hence, the right-hand side of Theorem 1 leads to

$$\begin{aligned} & \mathcal{M}_{h,k,l}^s \oplus \bigoplus_{i=1}^l \mathcal{M}_{h-i,k,l-i}^s \oplus \bigoplus_{i=1}^{k-l} \mathcal{M}_{h-i,k-i,l}^s \oplus \bigoplus_{i=1}^{k-l} \bigoplus_{j=1}^l \mathcal{M}_{h-i-j,k-i,l-j}^s \\ &= \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^l \mathcal{M}_{h-i-j,k-i,l-j}^s. \end{aligned}$$

This yields the following proposition.

Proposition 7. *For every $h > k \geq l$, the kernel space $\text{Ker}_h \mathcal{Q}_{k,l}$ for the invariant first-order operator $\mathcal{Q}_{k,l}$ decomposes as*

$$\begin{aligned} \text{Ker}_h \mathcal{Q}_{k,l} &= \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^l \mathcal{M}_{h-i-j,k-i,l-j}^s \\ &= \bigoplus_{i=0}^{k-l} \bigoplus_{j=0}^l \bigoplus_{p=0}^{k-i-l+j} \bigoplus_{q=0}^{l-j} (h-i-j+p+q, k-i-p, l-j-q)'. \end{aligned}$$

Proof. It suffices to verify that the dimensions of the left-hand side and the right-hand sides are equal, which is done using Maple and the results of Section 3. $\square$

Remark 4. *The module $\mathcal{S}_{h,1}$ appears twice in the decomposition of $\text{Ker}_h \mathcal{Q}_{2,1}$. On the one hand, it comes from*

$$\mathcal{M}_{h-1,1,1}^s = (h-1, 1, 1)' \oplus (h, 1)',$$

and on the other hand, we have

$$\mathcal{M}_{h-1,2} = (h-1, 2)' \oplus (h, 1)' \oplus (h+1)'. \quad \square$$

Also, in the decomposition of $\text{Ker}_h \mathcal{Q}_{3,1}$, there are 3 irreducible modules, $\mathcal{S}_{h+1,1}$, $\mathcal{S}_{h-1,1}$ and $\mathcal{S}_{h,2}$, which appear with multiplicity two. Let us also mention that the following 6 irreducible modules have multiplicity two in the decomposition of $\text{Ker}_h \mathcal{Q}_{4,1}$: $\mathcal{S}_{h+1,2}$, $\mathcal{S}_{h-1,2}$, $\mathcal{S}_{h,3}$, $\mathcal{S}_{h+2,1}$, $\mathcal{S}_{h,1}$ and $\mathcal{S}_{h-2,1}$. In [3] we determine, in general, which irreducible modules in $\text{Ker}_h \mathcal{Q}_{k,l}$ have multiplicity two or higher, and we prove that the corresponding embedding maps are linearly independent.

In order to obtain a compact expression for the dimension of $\mathcal{M}_{h,k,l}^s$, studied in [5], we apply Proposition 7 to $\text{Ker}_h \mathcal{Q}_{k,l}$, $\text{Ker}_{h-1} \mathcal{Q}_{k-1,l}$, $\text{Ker}_{h-1} \mathcal{Q}_{k,l-1}$ and $\text{Ker}_{h-2} \mathcal{Q}_{k-1,l-1}$ consecutively:

$$\text{Ker}_h \mathcal{Q}_{k,l} = \mathcal{M}_{h,k,l}^s \oplus \bigoplus_{i=1}^l \mathcal{M}_{h-i,k,l-i}^s \oplus \bigoplus_{i=1}^{k-l} \mathcal{M}_{h-i,k-i,l}^s \oplus \bigoplus_{i=1}^{k-l} \bigoplus_{j=1}^l \mathcal{M}_{h-i-j,k-i,l-j}^s,$$

$$\begin{aligned}
\text{Ker}_{h-1} \mathcal{Q}_{k-1,l} &= \bigoplus_{i=0}^{k-1-l} \bigoplus_{j=0}^l \mathcal{M}_{h-1-i-j, k-1-i, l-j}^s \\
&= \bigoplus_{i=1}^{k-l} \mathcal{M}_{h-i, k-i, l}^s \oplus \bigoplus_{i=1}^{k-l} \bigoplus_{j=1}^l \mathcal{M}_{h-i-j, k-i, l-j}^s, \\
\text{Ker}_{h-1} \mathcal{Q}_{k, l-1} &= \bigoplus_{i=0}^{k-l+1} \bigoplus_{j=0}^{l-1} \mathcal{M}_{h-1-i-j, k-i, l-1-j}^s \\
&= \bigoplus_{j=1}^l \mathcal{M}_{h-j, k, l-j}^s \oplus \bigoplus_{i=1}^{k-l+1} \bigoplus_{j=1}^l \mathcal{M}_{h-i-j, k-i, l-j}^s
\end{aligned}$$

and

$$\text{Ker}_{h-2} \mathcal{Q}_{k-1, l-1} = \bigoplus_{i=1}^{k-l+1} \bigoplus_{j=1}^l \mathcal{M}_{h-i-j, k-i, l-j}^s.$$

This leads to

Proposition 8. *The dimension of $\mathcal{M}_{h,k,l}^s$ is given by*

$$\begin{aligned}
&\frac{k(l-1)}{(k-l+1)(2n-1)(2n-2)} \dim_{\mathbb{C}} \mathcal{M}_{h,k,l}^s \\
&= \binom{h+2n-1}{h} \binom{k+2n-2}{k-1} \binom{l+2n-3}{l-2} \\
&\quad \cdot \frac{(2n+k+l-1)}{(k+1)l} \\
&\quad - \binom{h+2n-2}{h-1} \binom{k+2n-3}{k-2} \binom{l+2n-4}{l-3} \\
&\quad \cdot \frac{(2n+k+l-2)}{(k-1)(k+1)(l-2)l} (2kl+2nk+2nl-l-3k) \\
&\quad + \binom{h+2n-3}{h-2} \binom{k+2n-3}{k-2} \binom{l+2n-4}{l-3} \\
&\quad \cdot \frac{(2n+k+l-3)}{(k-1)(l-2)}.
\end{aligned}$$

Proof. The reasoning followed above, together with the generalized CK-extension (4),

leads to

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{M}_{h,k,l}^s &= \dim_{\mathbb{C}} \text{Ker}_h \mathcal{Q}_{k,l} - \dim_{\mathbb{C}} \text{Ker}_{h-1} \mathcal{Q}_{k-1,l} \\
&\quad - \dim_{\mathbb{C}} \text{Ker}_{h-1} \mathcal{Q}_{k,l-1} + \dim_{\mathbb{C}} \text{Ker}_{h-2} \mathcal{Q}_{k-1,l-1} \\
&= \binom{h+2n-1}{h} \binom{k+2n-2}{k-1} \binom{l+2n-3}{l-2} \\
&\quad \cdot \frac{(2n+k+l-1)(k-l+1)(2n-1)(2n-2)}{(k+1)k(l-1)l} \\
&\quad - \binom{h+2n-2}{h-1} \binom{k+2n-3}{k-2} \binom{l+2n-3}{l-2} \\
&\quad \cdot \frac{(2n+k+l-2)(k-l)(2n-1)(2n-2)}{(k-1)k(l-1)l} \\
&\quad - \binom{h+2n-2}{h-1} \binom{k+2n-2}{k-1} \binom{l+2n-4}{l-3} \\
&\quad \cdot \frac{(2n+k+l-2)(k-l+2)(2n-1)(2n-2)}{(k+1)k(l-2)(l-1)} \\
&\quad + \binom{h+2n-3}{h-2} \binom{k+2n-3}{k-2} \binom{l+2n-4}{l-3} \\
&\quad \cdot \frac{(2n+k+l-3)(k-l+1)(2n-1)(2n-2)}{(k-1)k(l-2)(l-1)}. \tag{21}
\end{aligned}$$

This expression may be simplified by combining the second and third term, leading to the result stated in the proposition. Note that, by [5], we have

$$\dim_{\mathbb{C}} \mathcal{M}_{h,k,l}^s = \sum_{i=0}^{k-l} \sum_{j=0}^l \dim_{\mathbb{C}} \mathcal{S}_{h+i+j, k-i, l-j}. \tag{22}$$

and, again using Maple, we verified that indeed (22)=(21). $\square$

10 Examples

We conclude this paper by illustrating our results with two explicit examples. The first example concerns the decomposition of $\text{Ker}_h \mathcal{Q}_{3,1}$, which looks like

$$\begin{aligned}
&(h+2, 1, 1)' \quad (h, 1, 1)' \quad (h-2, 1, 1)' \\
&\quad (h+1, 2, 1)' \quad (h-1, 2, 1)' \\
&\quad \quad (h, 3, 1)' \\
&\quad (h+2)' \quad (h)' \quad (h-2)' \\
&(h+3, 1)' \quad (h+1, 1)' \quad (h-1, 1)' \quad (h-3, 1)' \\
&\quad (h+2, 2)' \quad (h, 2)' \quad (h-2, 2)' \\
&\quad \quad (h+1, 3)' \quad (h-1, 3)'
\end{aligned}$$

where the underlined modules have multiplicity 2. The second example is the decomposition of $\text{Ker}_h \mathcal{Q}_{4,2}$, which looks like

$$\begin{array}{ccccccc}
(h+2, 2, 2)' & (h, 2, 2)' & (h-2, 2, 2)' & & & & \\
(h+1, 3, 2)' & (h-1, 3, 2)' & & & & & \\
(h, 4, 2)' & & & & & & \\
\\
(h+2, 1, 1)' & (h, 1, 1)' & (h-2, 1, 1)' & & & & \\
(h+3, 2, 1)' & \underline{(h+1, 2, 1)'} & \underline{(h-1, 2, 1)'} & (h-3, 2, 1)' & & & \\
(h+2, 3, 1)' & \underline{(h, 3, 1)'} & (h-2, 3, 1)' & & & & \\
(h+1, 4, 1)' & (h-1, 4, 1)' & & & & & \\
\\
(h+2)' & (h)' & (h-2)' & & & & \\
(h+3, 1)' & \underline{(h+1, 1)'} & \underline{(h-1, 1)'} & (h-3, 1)' & & & \\
(h+4, 2)' & \underline{(h+2, 2)'} & \underline{\underline{(h, 2)'}} & \underline{(h-2, 2)'} & (h-4, 2)' & & \\
(h+3, 3)' & \underline{(h+1, 3)'} & \underline{(h-1, 3)'} & (h-3, 3)' & & & \\
(h+2, 4)' & (h, 4)' & (h-2, 4)' & & & &
\end{array}$$

Here the modules which are underlined once appear twice in the decomposition, whereas the irreducible $\text{Spin}(m)$ -module with highest weight $(h, 2)'$, doubly underlined, has multiplicity 3. In [4] we explain in general the multiplicities and the geometry of the irreducibles appearing in the kernel space.

Finally it is noteworthy to remark that, by Proposition 7, $\text{Ker}_h \mathcal{Q}_{k,l}$ is constructed by means of the type A solutions corresponding to every irreducible $\text{Spin}(m)$ -module situated in the rectangle shown in the diagram below.

In [14] it is shown that this remarkable fact may be explained in terms of a non-trivial sequence between certain twistor operators.

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