On geometric SDPS-sets of elliptic dual polar spaces

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Abstract

Let \( n \in \mathbb{N} \setminus \{0, 1\} \) and let \( \mathbb{K} \) and \( \mathbb{K}' \) be fields such that \( \mathbb{K}' \) is a quadratic Galois extension of \( \mathbb{K} \). Let \( Q^{-}(2n+1, \mathbb{K}) \) be a nonsingular quadric of Witt index \( n \) in \( \text{PG}(2n+1, \mathbb{K}) \) whose associated quadratic form defines a nonsingular quadric \( Q^{+}(2n+1, \mathbb{K}') \) of Witt index \( n+1 \) in \( \text{PG}(2n+1, \mathbb{K}') \). For even \( n \), we define a class of SDPS-sets of the dual polar space \( DQ^{-}(2n+1, \mathbb{K}) \) associated to \( Q^{-}(2n+1, \mathbb{K}) \), and call its members geometric SDPS-sets. We show that geometric SDPS-sets of \( DQ^{-}(2n+1, \mathbb{K}) \) are unique up to isomorphism and that they all arise from the spin embedding of \( DQ^{-}(2n+1, \mathbb{K}) \). We will use geometric SDPS-sets to describe the structure of the natural embedding of \( DQ^{-}(2n+1, \mathbb{K}) \) into one of the half-spin geometries for \( Q^{+}(2n+1, \mathbb{K}') \).

Keywords: dual polar space, half-spin geometry, SDPS-set, spin embedding, hyperplane, valuation

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1 Introduction

Let \( n \in \mathbb{N} \setminus \{0, 1\} \), let \( \mathbb{K} \) and \( \mathbb{K}' \) be fields such that \( \mathbb{K}' \) is a quadratic Galois extension of \( \mathbb{K} \) and let \( \theta \) denote the unique nontrivial element in \( \text{Gal}(\mathbb{K}'/\mathbb{K}) \). Let \( Q^{-}(2n+1, \mathbb{K}) \) be a nonsingular quadric of Witt index \( n \) in \( \text{PG}(2n+1, \mathbb{K}) \) whose associated quadratic form defines a nonsingular quadric \( Q^{+}(2n+1, \mathbb{K}') \) of Witt index \( n+1 \) in \( \text{PG}(2n+1, \mathbb{K}') \). Let \( \mathcal{M}^{+} \) and \( \mathcal{M}^{-} \) denote the two systems of generators (= maximal subspaces) of \( Q^{+}(2n+1, \mathbb{K}') \). Recall that

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two generators belong to the same system if they intersect in a subspace of even co-dimension. For every $\epsilon \in \{+, -\}$, let $HS^\epsilon(2n + 1, K')$ denote the point-line geometry whose points are the elements of $\mathcal{M}^\epsilon$ and whose lines are the $(n - 2)$-dimensional subspaces of $Q^+(2n + 1, K')$ (natural incidence). The isomorphic geometries $HS^+(2n + 1, K')$ and $HS^-(2n + 1, K')$ are called the half-spin geometries for $Q^+(2n + 1, K')$. Let $DQ^-(2n + 1, K)$ denote the dual polar space associated to the quadric $Q^-(2n + 1, K)$. The map which associates with every generator of $Q^-(2n + 1, K)$ the unique element of $\mathcal{M}^\epsilon$ containing it, defines a full embedding of $DQ^-(2n + 1, K)$ into $HS^\epsilon(2n + 1, K')$, see Cooperstein and Shult [6] (for the finite case) and De Bruyn [9] (general case). This full embedding is called the natural embedding of $DQ^-(2n + 1, K)$ into $HS^\epsilon(2n + 1, K')$.

An SDPS-set of a dual polar space $\Delta$ of rank $2n'$ is a very nice set of points of $\Delta$ carrying the structure of a dual polar space of rank $n'$ (see Section 2). SDPS-sets of dual polar spaces were introduced by De Bruyn and Vandecasteele [11] because of their connection with the theory of valuations of near polygons. From that connection, it follows that the set of points of $\Delta$ at non-maximal distance from a given SDPS-set $X$ is a hyperplane of $\Delta$. We call this hyperplane the hyperplane of $\Delta$ associated to $X$.

In Section 4, we will construct a certain class of SDPS-sets of $DQ^-(2n + 1, K)$, $n$ even. The construction is as follows. Let $\alpha$ be a generator of $Q^+(2n + 1, K')$ which is disjoint from its conjugate $\alpha^\theta$ (with respect to the quadratic extension $K'$ of $K$). Let $H$ denote the following set of points of $\alpha$: a point $x$ of $\alpha$ belongs to $H$ if and only if $x$ is collinear on $Q^+(2n + 1, K')$ with its conjugate $x^\theta$. Then $H$ is a nonsingular Hermitian variety of Witt index $n/2$ of $\alpha$.

**Theorem 1.1** If $\beta$ is a generator of $H$, then $\langle \beta, \beta^\theta \rangle \cap PG(2n + 1, K)$ is a generator of $Q^-(2n + 1, K)$. The set of generators of $Q^-(2n + 1, K)$ which can be obtained in this way is an SDPS-set of $DQ^-(2n + 1, K)$.

Any SDPS-set of $DQ^-(2n + 1, K)$, $n$ even, which can be obtained as described in Theorem 1.1 is called geometric. We prove the following in Section 4.

**Theorem 1.2** Up to isomorphism, there exists a unique geometric SDPS-set in $DQ^-(2n + 1, K)$, $n$ even and $n \geq 2$.

The following theorem provides information regarding the structure of the natural embedding of $DQ^-(2n + 1, K)$ into one of the half-spin geometries for $Q^+(2n + 1, K')$. We will prove it in Section 5.
Theorem 1.3. Consider the natural embedding of \( \Delta = DQ^{-}(2n+1, \mathbb{K}) \) into \( HS^{\epsilon}(2n+1, \mathbb{K}') \), \( \epsilon \in \{+,-\} \). Let \( d_{\epsilon}(\cdot,\cdot) \) and \( d_{\Delta}(\cdot,\cdot) \) denote the distance functions in the respective geometries \( HS^{\epsilon}(2n+1, \mathbb{K}') \) and \( \Delta \). Then for every point \( x \) of \( HS^{\epsilon}(2n+1, \mathbb{K}') \), there exists a \( K \in \mathbb{N} \) and a geometric SDPS-set \( X \) in a convex subspace of diameter \( 2K \) of \( DQ^{-}(2n+1, \mathbb{K}) \) such that \( d_{\epsilon}(x,y) = \lceil \frac{K+1+d_{\Delta}(X,y)}{2} \rceil \) for every point \( y \) of \( \Delta \).

By [6] and [9], the dual polar space \( DQ^{-}(2n+1, \mathbb{K}) \) has a nice full embedding \( e \) into the projective space \( PG(2^n-1, \mathbb{K}') \), called the spin embedding of \( DQ^{-}(2n+1, \mathbb{K}) \). If \( \pi \) is a hyperplane of \( PG(2^n-1, \mathbb{K}') \), then the set of all points \( x \) of \( DQ^{-}(2n+1, \mathbb{K}) \) for which \( e(x) \in \pi \) is a hyperplane of \( DQ^{-}(2n+1, \mathbb{K}) \). Hyperplanes of \( DQ^{-}(2n+1, \mathbb{K}) \) which can be obtained in this way are said to arise from \( e \). In Section 5, we will also prove the following result.

Theorem 1.4. The hyperplanes of \( DQ^{-}(2n+1, \mathbb{K}) \), \( n \) even, associated to geometric SDPS-sets arise from the spin embedding of \( DQ^{-}(2n+1, \mathbb{K}) \).

Remark. An SDPS-set of \( DQ^{-}(5, \mathbb{K}) \) is nothing else than an ovoid of the generalized quadrangle \( DQ^{-}(5, \mathbb{K}) \). For any field \( \mathbb{K} \), there are ovoids in \( DQ^{-}(5, \mathbb{K}) \) which do not arise from the spin embedding, see e.g. Payne & Thas [14, p. 57] for the finite case and De Bruyn & Cardinali [4, Theorem 1.7] for the infinite case. So, an SDPS-set of \( DQ^{-}(5, \mathbb{K}) \) is not always geometric. It is still an open problem whether every SDPS-set of \( DQ^{-}(4m+1, \mathbb{K}), m \geq 2 \), is geometric.

2 Preliminaries

A near polygon is a partial linear space \( S = (\mathcal{P}, \mathcal{L}, I) \), \( I \subseteq \mathcal{P} \times \mathcal{L} \), with the property that for every point \( x \in \mathcal{P} \) and every line \( L \in \mathcal{L} \), there exists a unique point on \( L \) nearest to \( x \). Here, distances are measured in the collinearity graph \( \Gamma \) of \( S \). If \( d \) is the diameter of \( \Gamma \), then the near polygon is called a near 2d-gon. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles.

If \( S = (\mathcal{P}, \mathcal{L}, I) \) is a near polygon, then the distance between two points \( x \) and \( y \) of \( S \) will be denoted by \( d(x,y) \). The set of points at distance \( i \in \mathbb{N} \) from a given point \( x \in \mathcal{P} \) will be denoted by \( \Gamma_i(x) \). If \( x \in \mathcal{P} \) and \( \emptyset \neq X \subseteq \mathcal{P} \), then \( d(x,X) := \min\{d(x,y) \mid y \in X\} \).

A subspace \( S \) of a near polygon \( S \) is called convex if every point on a shortest path between two points of \( S \) is also contained in \( S \). The points
and lines contained in a convex subspace of $S$ define a sub-near-polygon of $S$. Convex subspaces of diameter $d'$ are therefore also called convex sub-2$d'$-gons. A convex subspace $F$ of $S$ is called classical in $S$ if for every point $x$ of $S$, there exists a necessarily unique point $\pi_F(x)$ in $F$ such that $d(x, y) = d(x, \pi_F(x)) + d(\pi_F(x), y)$ for every point $y$ of $F$.

A near polygon is called dense if every line is incident with at least three points and if every two points at distance 2 have at least 2 common neighbours. If $x$ and $y$ are two points of a dense near $2d$-gon at distance $d' \in \{0, \ldots, d\}$ from each other, then by Theorem 4 of Brouwer and Wilbrink [1], $x$ and $y$ are contained in a unique convex subspace $(x, y)$ of diameter $d'$. These convex subspaces are called quads if $d' = 2$, hexes if $d' = 3$ and maxes if $d' = d - 1$.

A function $f$ from the point-set of a dense near $2n$-gon $S$ to $\mathbb{N}$ is called a valuation of $S$ if it satisfies the following properties:

(V1) $f^{-1}(0) \neq \emptyset$;

(V2) every line $L$ of $S$ contains a necessarily unique point $x_L$ such that $f(x) = f(x_L) + 1$ for every point $x \in L \setminus \{x_L\}$;

(V3) every point $x$ of $S$ is contained in a necessarily unique convex subspace $F_x$ such that the following properties are satisfied for every $y \in F_x$: (i) $f(y) \leq f(x)$; (ii) if $z$ is a point collinear with $y$ such that $f(z) = f(y) - 1$, then $z \in F_x$.

Valuations of dense near polygons were introduced in De Bruyn and Vandecasteele [10]. We describe three constructions for obtaining valuations of a given dense near polygon $S = (P, L, I)$.

1. For every point $x$ of $S$, the map $f_x : P \rightarrow \mathbb{N}; y \mapsto d(x, y)$ is a valuation of $S$. We call $f_x$ a classical valuation of $S$.

2. Suppose $O$ is an ovoid of $S$, i.e. a set of points of $S$ meeting each line in a unique point. For every point $x$ of $S$, we define $f_O(x) := 0$ if $x \in O$ and $f_O(x) := 1$ otherwise. Then $f_O$ is a valuation of $S$. We call $f_O$ an ovoidal valuation of $S$.

3. Let $F = (P', L', I')$ be a convex sub-near-polygon of $S$ which is classical in $S$. Suppose that $f' : P' \rightarrow \mathbb{N}$ is a valuation of $F$. Then the map $f : P \rightarrow \mathbb{N}; x \mapsto f(x) := d(x, \pi_F(x)) + f'(\pi_F(x))$ is a valuation of $S$. We call $f$ the extension of $f'$. If $F = S$, then the extension is called trivial.

Valuations can also induce others.

**Proposition 2.1 ([10, Proposition 2.12])** Let $f$ be a valuation of a dense near polygon $S$, let $F$ be a convex subspace of $S$ and let $m$ denote the minimal
value attained by \( f(x) \) as \( x \) ranges over all points of \( F \). For every point \( x \) of \( F \), we define \( f_F(x) := f(x) - m \). Then \( f_F \) is a valuation of \( F \).

The valuation \( f_F \) defined in Proposition 2.1 is called the valuation of \( F \) induced by \( f \).

We now describe an important class of near polygons. Let \( \Pi \) be a nondegenerate polar space (Veblen and Kipling [18]; Tits [17, Chapter 7]) of rank \( n \geq 2 \). With \( \Pi \) there is associated a point-line geometry \( \Delta \) whose points are the maximal singular subspaces of \( \Pi \), whose lines are the next-to-maximal singular subspaces of \( \Pi \) and whose incidence relation is reverse containment. The geometry \( \Delta \) is called a dual polar space of rank \( n \) and is an example of a near \( 2n \)-gon (Cameron [3]). There exists a bijective correspondence between the nonempty convex subspaces of \( \Delta \) and the possibly empty singular subspaces of \( \Pi \). If \( \alpha \) is a singular subspace of \( \Pi \), then the set of all maximal singular subspaces of \( \Pi \) containing \( \alpha \) is a convex subspace of \( \Delta \). Conversely, every convex subspace of \( \Delta \) is obtained in this way. Every convex subspace of \( \Delta \) is classical in \( \Delta \). The point-line geometry induced on a convex subspace of diameter \( n' \geq 2 \) of \( \Delta \) is a dual polar space of rank \( n' \). If \( \alpha_1 \) and \( \alpha_2 \) are two maximal singular subspaces of \( \Pi \), then the distance between \( \alpha_1 \) and \( \alpha_2 \) in the dual polar space \( \Delta \) is equal to \( n - 1 - \dim (\alpha_1 \cap \alpha_2) \).

In the present paper, we will meet 3 classes of (dual) polar spaces. Let \( n \geq 2 \), let \( \mathbb{K} \) and \( \mathbb{K}' \) be two fields such that \( \mathbb{K}' \) is a quadratic Galois extension of \( \mathbb{K} \) and let \( \theta \) be the unique nontrivial element in \( \text{Gal}(\mathbb{K}'/\mathbb{K}) \).

(I) We denote by \( Q^- (2n + 1, \mathbb{K}) \) a nonsingular quadric of Witt index \( n \) in \( \text{PG}(2n + 1, \mathbb{K}) \) whose associated quadratic form defines a nonsingular quadric \( Q^+(2n + 1, \mathbb{K}') \) of Witt index \( n + 1 \) in \( \text{PG}(2n + 1, \mathbb{K}') \). With respect to a suitable reference system in \( \text{PG}(2n + 1, \mathbb{K}) \), \( Q^- (2n + 1, \mathbb{K}) \) has equation \( X_0^2 + (\delta + \delta^0)X_0X_1 + \delta^0X_1^2 + X_2X_3 + \cdots + X_{2n}X_{2n+1} = 0 \), where \( \delta \) is some element of \( \mathbb{K}' \setminus \mathbb{K} \). We denote by \( DQ^- (2n + 1, \mathbb{K}) \) and \( DQ^+(2n + 1, \mathbb{K}') \) the dual polar spaces associated to \( Q^- (2n + 1, \mathbb{K}) \) and \( Q^+(2n + 1, \mathbb{K}') \), respectively. We will call \((D)Q^- (2n + 1, \mathbb{K})\) an elliptic (dual) polar space and \((D)Q^+(2n + 1, \mathbb{K}')\) a hyperbolic (dual) polar space. (Notice that we have extended this terminology from the finite case to the infinite case.)

(II) We denote by \( H(2n, \mathbb{K}', \theta) \) a nonsingular \( \theta \)-Hermitian variety of Witt index \( n \) in \( \text{PG}(2n, \mathbb{K}') \) and by \( DH(2n, \mathbb{K}', \theta) \) the dual polar space associated to \( H(2n, \mathbb{K}', \theta) \). (With \( \theta \)-Hermitian we mean that the associated involutory automorphism is equal to \( \theta \).) With respect to a suitable reference system in \( \text{PG}(2n, \mathbb{K}') \), \( H(2n, \mathbb{K}', \theta) \) has equation \( X_0^{\theta+1} + (X_1X_2^\theta + X_2X_3^\theta) + \cdots + (X_{2n-1}X_2^\theta + X_{2n}X_2^{\theta-1}) = 0 \).
A hyperplane of a partial linear space $S = (P, L, I)$ is a proper subspace meeting each line. A full (projective) embedding of $S$ is an injective mapping $e$ from $P$ to the point-set of a projective space $\Sigma$ satisfying (i) $\langle e(P) \rangle = \Sigma$; (ii) $e(L) := \{e(x) | x \in L\}$ is a line of $\Sigma$ for every line $L$ of $S$. If $e$ is a full embedding of $S$ and if $\pi$ is a hyperplane of $\Sigma$, then $e^{-1}(e(P) \cap \pi)$ is a hyperplane of $S$. We say that the hyperplane $e^{-1}(e(P) \cap \pi)$ arises from the embedding $e$. Let $Q^{-}(2n+1, K)$ and $Q^{+}(2n+1, K')$ be the quadrics as defined above and let $HS(2n+1, K')$ denote one of the half-spin geometries for $Q^{+}(2n+1, K')$ (as defined in the Introduction). The geometry $HS(2n+1, K')$ has a nice full embedding into $PG(2n−1, K')$, see Chevalley [5] or Buekenhout and Cameron [2]. We refer to this particular embedding as the spin embedding of $HS(2n+1, K')$. Taking in mind the natural embedding of $DQ^{-}(2n+1, K)$ into $HS(2n+1, K')$, we see that the spin embedding of $HS(2n+1, K')$ induces a full embedding of $DQ^{-}(2n+1, K)$ into a subspace $\Sigma$ of $PG(2n−1, K')$. It can be shown, see Cooperstein and Shult [6] and De Bruyn [9] that $\Sigma = PG(2n−1, K')$. The induced embedding of $DQ^{-}(2n+1, K)$ into $PG(2n−1, K')$ is called the spin embedding of $DQ^{-}(2n+1, K)$.

Let $\Delta$ be a thick dual polar space of rank $2n$. A set $X$ of points of $\Delta$ is called an SDPS-set of $\Delta$ if it satisfies the following properties:

(SDPS1) No two points of $X$ are collinear in $\Delta$.

(SDPS2) If $x, y \in X$ such that $d(x, y) = 2$, then $X \cap \langle x, y \rangle$ is an ovoid of the quad $\langle x, y \rangle$.

(SDPS3) The point-line geometry $\tilde{\Delta}$ whose points are the elements of $X$ and whose lines are the quads of $\Delta$ containing at least two points of $X$ (natural incidence) is a dual polar space of rank $n$.

(SDPS4) For all $x, y \in X$, $d(x, y) = 2 \cdot \tilde{d}(x, y)$, where $\tilde{d}(x, y)$ denotes the distance between $x$ and $y$ in the dual polar space $\tilde{\Delta}$.

(SDPS5) If $x \in X$ and $L$ is a line of $\Delta$ through $x$, then $L$ is contained in a (necessarily unique) quad of $\Delta$ which contains at least two points of $X$.

SDPS-sets of dual polar spaces were introduced in De Bruyn and Vandecasteele [11]. The discussion in [11] is however restricted to the finite case. For a discussion including the infinite case, see De Bruyn [7, Section 5.6.7]. SDPS-sets give rise to valuations:

**Proposition 2.2 (Theorem 5.29 of [7])** Let $X$ be an SDPS-set of a thick dual polar space $\Delta$ of rank $2n$. For every point $x$ of $\Delta$, we define $f(x) := d(x, X)$. Then $f$ is a valuation of $\Delta$ whose maximal value is equal to $n$. 

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A valuation which can be obtained from an SDPS-set in the way as described in Proposition 2.2 is called an SDPS-valuation. By Property (V2) in the definition of valuation, we have

**Corollary 2.3** Let $X$ be an SDPS-set of a thick dual polar space of rank $2n$. Let $H$ denote the set of points of $\Delta$ at distance at most $n - 1$ from $X$. Then $H$ is a hyperplane of $\Delta$ (the so-called hyperplane of $\Delta$ associated to $X$).

SDPS-valuations can be characterized as follows.

**Proposition 2.4** (Theorem 5.32 of [7]) Let $\Delta$ be a thick dual polar space and let $f$ be a valuation of $\Delta$ with the property that every induced hex valuation is either classical or the extension of an ovoidal valuation of a quad. Then $f$ is the (possibly trivial) extension of an SDPS-valuation of a convex subpolygon of $\Delta$.

### 3 Notations and basic lemmas

Let $K$ and $K'$ be fields such that $K'$ is a quadratic Galois extension of $K$. Let $\theta$ denote the unique nontrivial element in $\text{Gal}(K'/K)$ and let $n \in \mathbb{N} \setminus \{0, 1\}$.

Let $V(2n+2, K')$ denote a $(2n+2)$-dimensional vector space over the field $K'$ and suppose $B^* = \{e_0^*, e_1^*, \ldots, e_{2n+1}^*\}$ is a basis of $V(2n+2, K')$. The set of all $K$-linear combinations of elements of $B^*$ defines a $(2n+2)$-dimensional vector space $V(2n+2, K)$ over the field $K$. If $\bar{x} = \sum_{i=0}^{2n+1} X_i e_i^*$ is a vector of $V(2n+2, K')$, then we define $\bar{x}^\theta = \sum_{i=0}^{2n+1} X_i^\theta e_i^*$.

Let $\text{PG}(2n+1, K')$ and $\text{PG}(2n+1, K)$ denote the projective spaces associated to $V(2n+2, K')$ and $V(2n+2, K)$, respectively. An ordered basis $(\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n+1})$ of $V(2n+2, K')$ is called a reference system for $\text{PG}(2n+1, K)$ if $\sum_{i=0}^{2n+1} X_i \bar{e}_i \in \text{PG}(2n+1, K)$ for all $X_0, X_1, \ldots, X_{2n+1} \in K$ with $(X_0, X_1, \ldots, X_{2n+1}) \neq (0, 0, \ldots, 0)$. If $p = (\sum_{i=0}^{2n+1} X_i e_i^*)$ is a point of $\text{PG}(2n+1, K')$, then we define $p^\theta := (\sum_{i=0}^{2n+1} X_i^\theta e_i^*)$. For every subspace $\alpha$ of $\text{PG}(2n+1, K')$, we define $\alpha^\theta := \{p^\theta \mid p \in \alpha\}$. Notice that we have given different meanings to the map $\theta$, but from the context it will always be clear what is meant.

There is a natural inclusion of the projective space $\text{PG}(2n+1, K)$ into the projective space $\text{PG}(2n+1, K')$. In the sequel, we will regard points of $\text{PG}(2n+1, K)$ as points of $\text{PG}(2n+1, K')$. Every subspace $\alpha$ of $\text{PG}(2n+1, K')$ then generates a subspace $\alpha'$ of $\text{PG}(2n+1, K')$ of the same dimension as $\alpha$.

**Lemma 3.1** (Lemma 2.1 of [9]) If $\alpha$ is a subspace of $\text{PG}(2n+1, K')$, then there exists a unique subspace $\beta$ of $\text{PG}(2n+1, K')$ such that $\alpha \cap \alpha^\theta = \beta'$. 

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For all \(i, j \in \{0, \ldots, 2n+1\}\) with \(i \leq j\), let \(a_{ij} \in \mathbb{K}\) such that

\[
q\left( \sum_{i=0}^{2n+1} X_i \varepsilon_i^\theta \right) := \sum_{0 \leq i \leq j \leq 2n+1} a_{ij} X_i X_j
\]

is a quadratic form of \(V(2n+2, \mathbb{K})\) and \(V(2n+2, \mathbb{K}')\) defining a nonsingular quadric \(Q^{-}(2n+1, \mathbb{K})\) of Witt index \(n\) in \(\mathrm{PG}(2n+1, \mathbb{K})\) and a nonsingular quadric \(Q^{+}(2n+1, \mathbb{K}')\) of Witt index \(n+1\) in \(\mathrm{PG}(2n+1, \mathbb{K}')\). Let \(B(\cdot, \cdot)\) denote the bilinear form of \(V(2n+2, \mathbb{K}')\) associated to the quadratic form \(q(\cdot)\), i.e.

\[
B(\bar{x}_1, \bar{x}_2) = q(\bar{x}_1 + \bar{x}_2) - q(\bar{x}_1) - q(\bar{x}_2)
\]

for all \(\bar{x}_1, \bar{x}_2 \in V(2n+2, \mathbb{K}')\). Obviously, we have

\[
q(\bar{x}_1^\theta) = [q(\bar{x}_1)]^\theta,
\]

\[
B(\bar{x}_1^\theta, \bar{x}_2^\theta) = [B(\bar{x}_1, \bar{x}_2)]^\theta,
\]

for all \(\bar{x}_1, \bar{x}_2 \in V(2n+2, \mathbb{K}')\).

Let \(\mathcal{M}^+\) and \(\mathcal{M}^-\) denote the two systems of generators of \(Q^{+}(2n+1, \mathbb{K}')\) and put \(\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-\).

**Lemma 3.2 (Lemma 2.2 of [9])** We have \((\mathcal{M}^+)^\theta = \mathcal{M}^-\) and \((\mathcal{M}^-)^\theta = \mathcal{M}^+\). As a consequence, for every \(\alpha \in \mathcal{M}\), \(n - \dim(\alpha \cap \alpha^\theta)\) is odd.

**Lemma 3.3** Let \(k \in \{-1, 0, \ldots, n - 1\}\) such that \(n - k\) is odd. Then there exists an \(\alpha \in \mathcal{M}\) such that \(\dim(\alpha \cap \alpha^\theta) = k\).

**Proof.** We can choose a reference system \((\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n+1})\) for \(\mathrm{PG}(2n+1, \mathbb{K})\) and a \(\delta \in \mathbb{K}' \setminus \mathbb{K}\) in such a way that a point \(\left(\sum_{i=0}^{2n+1} X_i \bar{e}_i\right)\) of \(\mathrm{PG}(2n+1, \mathbb{K})\) belongs to \(Q^{-}(2n+1, \mathbb{K})\) if and only if

\[
X_0^2 + (\delta + \delta^\theta)X_0X_1 + \delta^{\theta+1}X_1^2 + X_2X_3 + \cdots + X_{2n}X_{2n+1} = 0.
\]

Now, let \(\alpha\) be the element of \(\mathcal{M}\) generated by the points \(\left\langle \delta \bar{e}_0 - \bar{e}_1 \right\rangle, \left\langle \bar{e}_{4i-2} + \delta \bar{e}_{4i} \right\rangle \quad (i \in \{1, \ldots, \frac{n-k-1}{2}\}), \left\langle \bar{e}_{4i-1} - \frac{1}{\delta} \bar{e}_{4i+1} \right\rangle \quad (i \in \{1, \ldots, \frac{n-k-1}{2}\}), \left\langle \bar{e}_{2n-2i} \right\rangle \quad (i \in \{0, \ldots, k\}).\) Then one readily verifies that \(\alpha \cap \alpha^\theta = \left\langle \bar{e}_{2n-2i} \mid 0 \leq i \leq k\right\rangle\). Hence, \(\dim(\alpha \cap \alpha^\theta) = k\). \(\blacksquare\)

**Remark.** Let \(\pi\) be a subspace of dimension \(k \in \{-1, 0, \ldots, n-3\}\) of \(Q^{-}(2n+1, \mathbb{K})\). The subspaces of \(Q^{-}(2n+1, \mathbb{K})\) through \(\pi\) define a polar space \(P\). The subspaces of \(Q^{+}(2n+1, \mathbb{K}')\) through \(\pi'\) define a polar space \(P'\). We can choose a \(\delta \in \mathbb{K}' \setminus \mathbb{K}\) and a reference system \((\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n+1})\) for \(\mathrm{PG}(2n+1, \mathbb{K})\)
such that (i) \( q\left( \sum_{i=0}^{2n+1} X_i e_i \right) = X_0^2 + (\delta + \delta^\theta)X_0X_1 + \delta^{\theta+1}X_1^2 + X_2X_3 + \cdots + X_{2n}X_{2n+1} \), (ii) \( \pi \) is the subspace of \( \text{PG}(2n+1, \mathbb{K}) \) corresponding to the subspace of \( V(2n+2, \mathbb{K}) \) generated by \( e_{2n+1}, e_{2n-1}, \ldots, e_{2n-2k} \). Let \( \tilde{\pi} \) denote the subspace of \( \text{PG}(2n+1, \mathbb{K}) \) defined by the vectors \( \bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n-2k} \). The quadratic form \( \tilde{q}\left( \sum_{i=0}^{2n-2k-1} X_i \bar{e}_i \right) = X_0^2 + (\delta + \delta^\theta)X_0X_1 + \delta^{\theta+1}X_1^2 + X_2X_3 + \cdots + X_{2n-2k-2}X_{2n-2k-1} \) defines a nonsingular quadric \( \tilde{Q}^{-}(2n - 2k - 1, \mathbb{K}) \) of Witt index \( n - k - 1 \) in \( \tilde{\pi} \) and a nonsingular quadric \( \tilde{Q}^{+}(2n - 2k - 1, \mathbb{K}') \) of Witt index \( n - k \) in \( \tilde{\pi}' \). There exists a natural bijection between the singular subspaces of \( P \) (respectively \( P' \)) and the subspaces contained in the quadric \( \tilde{Q}^{-}(2n - 2k - 1, \mathbb{K}) \) (respectively \( \tilde{Q}^{+}(2n - 2k - 1, \mathbb{K}') \)): if \( \alpha \) (respectively \( \alpha' \)) is a subspace of \( Q^{-}(2n + 1, \mathbb{K}) \) (respectively \( Q^{+}(2n + 1, \mathbb{K}') \)) through \( \pi \) (respectively \( \pi' \)), then \( \alpha \cap \pi \) (respectively \( \alpha' \cap \pi' \)) is a subspace of \( \tilde{Q}^{-}(2n - 2k - 1, \mathbb{K}) \) (respectively \( \tilde{Q}^{+}(2n - 2k - 1, \mathbb{K}') \)). Hence, \( P \cong Q^{-}(2n - 2k - 1, \mathbb{K}) \) and \( P' \cong Q^{+}(2n - 2k - 1, \mathbb{K}') \). Notice also that the elements of one system of generators of \( Q^{+}(2n + 1, \mathbb{K}') \) through \( \pi' \) define one system of generators of \( P' \cong \tilde{Q}^{+}(2n - 2k - 1, \mathbb{K}') \). We will freely make use of this remark in the sequel.

4 Geometric SDPS-sets of \( DQ^{-}(2n + 1, \mathbb{K}) \)

We will continue with the notation introduced in Section 3. In this section however, we will assume that \( n \) is even and that \( \alpha \) is an element of \( \mathcal{M} \) satisfying \( \alpha \cap \alpha^\theta = \emptyset \). By Lemma 3.3 we know that such an \( \alpha \) exists. Notice that also \( \alpha^\theta \in \mathcal{M} \) and \( \alpha \cap \text{PG}(2n+1, \mathbb{K}) = \emptyset \) since every point of \( \alpha \cap \text{PG}(2n + 1, \mathbb{K}) \) is contained in \( \alpha \cap \alpha^\theta \).

Lemma 4.1 For every subspace \( \beta \) of \( \alpha \), \( \gamma = \langle \beta, \beta^\theta \rangle \cap \text{PG}(2n+1, \mathbb{K}) \) is a subspace of \( \text{PG}(2n+1, \mathbb{K}) \) of dimension \( 2 \cdot \dim(\beta) + 1 \). Moreover, \( \gamma' = \langle \beta, \beta^\theta \rangle \).

Proof. Since \( \beta \subseteq \alpha \) and \( \beta^\theta \subseteq \alpha^\theta \) are disjoint, \( \langle \beta, \beta^\theta \rangle \) has dimension \( 2 \cdot \dim(\beta) + 1 \). Now, by Lemma 3.1, there exists a subspace \( \gamma_1 \) of \( \text{PG}(2n+1, \mathbb{K}) \) such that \( \gamma'_1 = \langle \beta, \beta^\theta \rangle \cap \langle \beta, \beta^\theta \rangle^\theta = \langle \beta, \beta^\theta \rangle \). Obviously, \( \dim(\gamma_1) = \dim(\langle \beta, \beta^\theta \rangle) = 2 \cdot \dim(\beta) + 1 \) and \( \gamma_1 = \langle \beta, \beta^\theta \rangle \cap \text{PG}(2n+1, \mathbb{K}) \). □

Now, let \( H \) denote the set of all points \( \langle \bar{x} \rangle \) of \( \alpha \) for which \( h(\bar{x}) := B(\bar{x}, \bar{x}^\theta) = 0 \). Obviously, \( H \) is a \( \theta \)-Hermitian variety of \( \alpha \). We observe the following for two points \( \langle \bar{x}, \bar{y} \rangle \) of \( \alpha \):

(I) \( \langle \bar{x} \rangle \) and \( \langle \bar{y}^\theta \rangle \) are collinear on the quadric \( Q^{+}(2n + 1, \mathbb{K}') \) if and only if \( B(\bar{x}, \bar{y}^\theta) = 0 \);
(II) if \( \langle \bar{x} \rangle \in H \) and \( \langle \bar{y} \rangle \neq \langle \bar{x} \rangle \), then \( B(\bar{x}, \bar{y}^\theta) = 0 \) if and only if the line of \( \alpha \) through \( \langle \bar{x} \rangle \) and \( \langle \bar{y} \rangle \) is either contained in \( H \) or intersects \( H \) in the point \( \langle \bar{x} \rangle \).

By (I), a point \( p \in \alpha \) belongs to \( H \) if and only if \( p \) is collinear on \( Q^+(2n+1, \K') \) with \( p^\theta \).

**Lemma 4.2** \( H \) is nonsingular.

**Proof.** Suppose \( \langle \bar{x} \rangle \) is a singular point of \( H \). Then by (II) above, \( B(\bar{x}, \bar{y}^\theta) = 0 \) for all \( \bar{y} \in V(2n+2, \K') \) such that \( \langle \bar{y} \rangle \) is a point of \( \alpha \). Hence, by (I) above, \( \langle \bar{x} \rangle \) is collinear on \( Q^+(2n+1, \K') \) with every point of \( \alpha^\theta \). This is impossible since \( \alpha^\theta \) is a generator of \( Q^+(2n+1, \K') \) and \( \langle \bar{x} \rangle \notin \alpha^\theta \). ■

**Lemma 4.3** If \( \beta \) is a subspace of \( \alpha \) contained in \( H \), then \( \langle \beta, \beta^\theta \rangle \cap PG(2n+1, \K) \) is a subspace of \( Q^-(2n+1, \K) \) of dimension \( 2 \cdot \dim(\beta) + 1 \).

**Proof.** Put \( k := \dim(\beta) + 1 \) and let \( \{p_1, p_2, \ldots, p_k\} \) be an independent generating set of points for the subspace \( \beta \). Then \( \{p_1, p_2, \ldots, p_k, p_1^\theta, p_2^\theta, \ldots, p_k^\theta\} \) is an independent generating set of points for the subspace \( \langle \beta, \beta^\theta \rangle \). Now, by (I) and (II) above, \( \{p_1, p_2, \ldots, p_k, p_1^\theta, p_2^\theta, \ldots, p_k^\theta\} \) is a set of mutually collinear points of the quadric \( Q^+(2n+1, \K') \). By Lemma 4.1, it now follows that \( \langle \beta, \beta^\theta \rangle \cap PG(2n+1, \K) \) is a subspace of dimension \( 2 \cdot \dim(\beta) + 1 \) of \( Q^-(2n+1, \K) \). ■

**Lemma 4.4** Let \( x \) be a point of \( PG(2n+1, \K) \). Then there exists a unique line \( L_x \) in \( PG(2n+1, \K') \) through \( x \) which meets \( \alpha \) and \( \alpha^\theta \) in points. Moreover, \( (L_x \cap \alpha)^\theta = L_x \cap \alpha^\theta \) and \( L_x \cap PG(2n+1, \K) \) is a line of \( PG(2n+1, \K) \). If \( x \in Q^-(2n+1, \K) \), then \( L_x \subseteq Q^+(2n+1, \K') \) and \( L_x \cap PG(2n+1, \K) \) is a line of \( Q^-(2n+1, \K) \).

**Proof.** Clearly, there is a unique line \( L_x \) through \( x \) meeting \( \alpha \) and \( \alpha^\theta \) in points, namely the line through the points \( \langle \alpha, x \rangle \cap \alpha^\theta \) and \( \langle \alpha^\theta, x \rangle \cap \alpha \). Since \( L_x \) meets \( \alpha \) and \( \alpha^\theta \) and contains the point \( x \), also \( L_x^\theta \) meets \( \alpha \) and \( \alpha^\theta \) and contains the point \( x^\theta = x \). Hence, \( L_x^\theta = L_x \). This implies that \( (L_x \cap \alpha)^\theta = L_x \cap \alpha^\theta \).

By Lemma 4.1, \( L_x \cap PG(2n+1, \K) \) is a line of \( PG(2n+1, \K) \).

Suppose now that \( x \in Q^-(2n+1, \K) \). Then the line \( L_x \) contains three points of \( Q^+(2n+1, \K') \), namely the point \( x \) and the unique points in \( L_x \cap \alpha \) and \( L_x \cap \alpha^\theta \). Hence, \( L_x \subseteq Q^+(2n+1, \K') \). It follows that \( L_x \cap PG(2n+1, \K) \) is a line of \( Q^-(2n+1, \K) \). ■

**Lemma 4.5** Let \( \beta \) be a subspace of \( \alpha \) contained in \( H \) and let \( \gamma \) be the subspace \( \langle \beta, \beta^\theta \rangle \cap PG(2n+1, \K) \) of \( Q^-(2n+1, \K) \). Let \( x \) be a point of
$Q^-(2n+1, \mathbb{K}) \setminus \gamma$ which is collinear on $Q^-(2n+1, \mathbb{K})$ with every point of $\gamma$, and let $L_x$ denote the unique line of $\text{PG}(2n+1, \mathbb{K}')$ through $x$ which meets $\alpha$ and $\alpha^\theta$ in the respective points $v$ and $v^\theta$. Then

(i) $L_x$ and $\langle \beta, v \rangle$ are disjoint;
(ii) the subspace $\langle \beta, v \rangle$ of $\alpha$ is contained in $H$.

**Proof.** (i) Since $x \notin \gamma$, also $x \notin \langle \beta, \beta^\theta \rangle$. Suppose $L_x \cap \langle \beta, \beta^\theta \rangle$ is a singleton \{y\}. By Lemma 4.4, $L_x$ is generated by a line of $\text{PG}(2n+1, \mathbb{K})$ which is contained in $Q^-(2n+1, \mathbb{K})$. Since both $L_x$ and $\langle \beta, \beta^\theta \rangle = \gamma'$ are generated by subspaces of $\text{PG}(2n+1, \mathbb{K})$, the point $y$ must belong to $\text{PG}(2n+1, \mathbb{K})$. Since $y \in \langle \beta, \beta^\theta \rangle \setminus (\beta \cup \beta^\theta)$, there exists a unique line through $y$ meeting $\beta$ and $\beta^\theta$ and this line necessarily coincides with the unique line through $y$ meeting $\alpha$ and $\alpha^\theta$. It follows that $L_x$ meets $\beta$ and $\beta^\theta$, contradicting the fact that $L_x$ is not contained in $\langle \beta, \beta^\theta \rangle$ (recall $x \notin \langle \beta, \beta^\theta \rangle$). Hence, $L_x$ and $\langle \beta, \beta^\theta \rangle$ are disjoint.

(ii) We have $\beta \subseteq H$. Since $L_x \subseteq Q^+(2n+1, \mathbb{K}')$, $v$ and $v^\theta$ are collinear on $Q^+(2n+1, \mathbb{K}')$, i.e. $v \in H$. In order to show that $\langle \beta, v \rangle \subseteq H$, we need to prove that every point $u$ of $\beta$ is collinear on $H$ with $v$, or equivalently, that every point $u$ of $\beta$ is collinear with $v^\theta$ on the quadric $Q^+(2n+1, \mathbb{K}')$ (see (I) and (II) above).

Since $x$ is collinear on $Q^-(2n+1, \mathbb{K})$ with every point of $\gamma$, it is collinear on $Q^+(2n+1, \mathbb{K}')$ with every point of $\gamma' = \langle \beta, \beta^\theta \rangle$. In particular, $x$ is collinear on $Q^+(2n+1, \mathbb{K}')$ with $u$. Now, since $u$ is collinear on $Q^+(2n+1, \mathbb{K}')$ with $v$ and $x$, it is also collinear on $Q^+(2n+1, \mathbb{K}')$ with $v^\theta$. This is precisely what we needed to show.

**Proposition 4.6** $H$ is a nonsingular $\theta$-Hermitian variety of (maximal) Witt index $\frac{n}{2}$ in $\alpha$.

**Proof.** In view of Lemma 4.2, we need to show that there exists an $(\frac{n}{2} - 1)$-dimensional subspace on $H$.

We prove by induction on $k \in \{0, \ldots, \frac{n}{2}\}$ that there exists a subspace $\beta_k$ of dimension $k - 1$ on $H$. Obviously, this claim holds if $k = 0$. So, suppose $k \geq 1$. By the induction hypothesis, there exists a subspace $\beta_{k-1}$ of dimension $k-2$ on $H$. Put $\gamma_{k-1} := \langle \beta_{k-1}, \beta_{k-1}^\theta \rangle \cap \text{PG}(2n+1, \mathbb{K})$. By Lemma 4.3, $\gamma_{k-1}$ is a subspace of dimension $2k - 3$ of $Q^-(2n+1, \mathbb{K})$. Since $k \leq \frac{n}{2}$, there exists a point $u_k \in Q^-(2n+1, \mathbb{K})$ which is collinear on $Q^-(2n+1, \mathbb{K})$ with every point of $\gamma_{k-1}$. Let $L_{u_k}$ denote the unique line through $u_k$ meeting $\alpha$ and $\alpha^\theta$ in the respective points $v_k$ and $v_k^\theta$ (see Lemma 4.4). By Lemma 4.5, $\beta_k := \langle \beta_{k-1}, v_k \rangle \subseteq H$ and $\dim(\beta_k) = k - 1$. \[\square\]
Proposition 4.7 Let $X$ be the set of generators of $Q^-(2n+1, \mathbb{K})$ of the form $\langle \beta, \beta^\theta \rangle \cap \PG(2n+1, \mathbb{K})$, where $\beta$ is some generator of $H$. Then $X$ is an SDPS-set of the dual polar space $DQ^-(2n+1, \mathbb{K})$. Moreover, the dual polar space defined on the set $X$ by the quads of $DQ^-(2n+1, \mathbb{K})$ containing at least two points of $X$ is isomorphic to the dual polar space associated to $H$.

Proof. Let $d(\cdot, \cdot)$ denote the distance function in the dual polar space $DQ^-(2n+1, \mathbb{K})$. Let $DH(n, \mathbb{K}', \theta)$ denote the dual polar space associated to $H = H(n, \mathbb{K}', \theta)$ and let $d'(\cdot, \cdot)$ denote the distance function in $DH(n, \mathbb{K}', \theta)$.

For every subspace $\gamma$ of $H$, we define $\gamma^\phi := \langle \gamma, \gamma^\theta \rangle \cap \PG(2n+1, \mathbb{K})$. By Lemma 4.3, $\gamma^\phi$ is a subspace of $Q^-(2n+1, \mathbb{K})$ of dimension $2 \cdot \dim(\gamma) + 1$. So, if $\gamma$ is a point of $DH(n, \mathbb{K}', \theta)$, then $\gamma^\phi$ is a point of $DQ^-(2n+1, \mathbb{K})$. If $\gamma_1$ and $\gamma_2$ are two distinct subspaces on $X$, then $\gamma_1^\phi \cap \gamma_2^\phi = \langle \gamma_1, \gamma_1^\theta \rangle \cap \langle \gamma_2, \gamma_2^\phi \rangle \cap \PG(2n+1, \mathbb{K}) = \langle \gamma_1 \cap \gamma_2, (\gamma_1 \cap \gamma_2)^\theta \rangle \cap \PG(2n+1, \mathbb{K}) = (\gamma_1 \cap \gamma_2)^\phi$. Hence,

$$d(\beta_1^\phi, \beta_2^\phi) = 2 \cdot d'(\beta_1, \beta_2)$$

for any two points $\beta_1$ and $\beta_2$ of $DH(n, \mathbb{K}', \theta)$. This proves property (SDPS1).

It is also obvious that $\phi$ defines a bijection between the set of lines of $DH(n, \mathbb{K}', \theta)$ and the set of quads of $DQ^-(2n+1, \mathbb{K})$ which contain at least two points of $X$. As a consequence, the partial linear space $\tilde{\Delta}$ whose points are the elements of $X$ and whose lines are the quads of $DQ^-(2n+1, \mathbb{K})$ containing at least two points of $X$ (natural incidence) is isomorphic to $DH(n, \mathbb{K}', \theta)$, proving property (SDPS3). Property (SDPS4) now immediately follows from equation (1).

We now prove property (SDPS2). Let $\beta_1$ be a line of $DH(n, \mathbb{K}', \theta)$ and put $\gamma_1 := \langle \beta_1, \beta_1^\theta \rangle \cap \PG(2n+1, \mathbb{K}) \subseteq Q^-(2n+1, \mathbb{K})$. Let $\gamma_2$ be an arbitrary subspace of dimension $n-2$ of $Q^-(2n+1, \mathbb{K})$ containing $\gamma_1$. We need to prove that there exists a unique generator $\beta_2$ of $H(n, \mathbb{K}', \theta)$ through $\beta_1$ such that $\gamma_2 \subseteq \langle \beta_2, \beta_2^\theta \rangle \cap \PG(2n+1, \mathbb{K})$. Let $x$ be an arbitrary point of $\gamma_2 \backslash \gamma_1$ and let $L_x$ denote the unique line through $x$ meeting $\alpha$ and $\alpha^\theta$ in the respective points $v$ and $v^\theta$. By Lemma 4.5, $L_x \cap \langle \beta_1, \beta_1^\theta \rangle = \emptyset$ and $v$ is collinear on the Hermitian variety $H$ with every point of $\beta_1$. If we put $\beta^* := \langle \beta_1, v \rangle$, then $\beta^*$ is a generator of $H(n, \mathbb{K}', \theta)$ through $\beta_1$ satisfying $\gamma_2 \subseteq \langle \beta^*, (\beta^*)^\theta \rangle \cap \PG(2n+1, \mathbb{K})$. Conversely, suppose that $\beta_2$ is a generator of $H(n, \mathbb{K}', \theta)$ through $\beta_1$ such that $\gamma_2 \subseteq \langle \beta_2, \beta_2^\theta \rangle \cap \PG(2n+1, \mathbb{K})$. Since $x \in \langle \beta_2, \beta_2^\theta \rangle \backslash (\beta_2 \cup \beta_2^\theta)$, there exists a unique line through $x$ meeting $\beta_2$ and $\beta_2^\theta$. This line necessarily coincides with $L_x$. Hence, $v \in \beta_2$ and $\beta_2 = \langle \beta_1, v \rangle = \beta^*$. Property (SDPS2) immediately follows.

We now prove property (SDPS5). Let $\gamma_1$ be a generator of $Q^-(2n+1, \mathbb{K})$ corresponding to a point of $X$ and let $\gamma_2$ be an arbitrary hyperplane of $\gamma_1$. The proof proceeds similarly, but considering the dual polar space $DQ^-(2n+1, \mathbb{K})$ instead of $DQ^+(2n+1, \mathbb{K})$.
There exists a unique generator $\beta_1$ of $H$ such that $\langle \beta_1, \beta_0^\theta \rangle \cap \PG(2n+1, \mathbb{K}) = \gamma_1$. Now, $\gamma'_2$ is a hyperplane of $\gamma'_1 = \langle \beta_1, \beta_0^\theta \rangle$ and hence intersects $\beta_1$ in either $\beta_1$ or a hyperplane of $\beta_1$. Suppose $\beta_1 \subseteq \gamma'_2$. Then $\beta_1 \subseteq \gamma'_2 = \gamma'_2$ and hence $\langle \beta_1, \beta_0^\theta \rangle \subseteq \gamma'_2$, a contradiction. Hence, $\gamma'_2$ intersects $\beta_1$ in a hyperplane $\beta_2$ of $\beta_1$. Since $\beta_2 \subseteq \gamma'_2$, we have $\beta_2 \subseteq \gamma'_2 = \gamma'_2$, $\langle \beta_2, \beta_0^\theta \rangle \subseteq \gamma'_2$ and hence $\langle \beta_2, \beta_0^\theta \rangle \cap \PG(2n+1, \mathbb{K}) \subseteq \gamma'_2 \cap \PG(2n+1, \mathbb{K}) = \gamma_2$. So, the $(n-3)$-dimensional subspace $\langle \beta_2, \beta_0^\theta \rangle \cap \PG(2n+1, \mathbb{K})$ of $Q^-(2n+1, \mathbb{K})$ corresponds to a quad of $DQ^-(2n+1, \mathbb{K})$ which contains the line of $DQ^-(2n+1, \mathbb{K})$ corresponding to $\gamma_2$. This proves property (SDPS5).

SDPS-sets of the dual polar space $DQ^-(2n+1, \mathbb{K})$ which can be obtained as described in Proposition 4.7 are called geometric SDPS-sets of the dual polar space $DQ^-(2n+1, \mathbb{K})$. A certain class of SDPS-sets of $DQ^-(2n+1, \mathbb{K})$ has already been described in De Bruyn & Vandecasteele [11] and Pralle & Shpectorov [15]. All these SDPS-sets are geometric. We will prove this in the appendix of this paper using the description of [11].

**Definition.** Again, suppose that $n$ is even and consider the inclusion $\PG(n-1, \mathbb{K}) \subset \PG(n-1, \mathbb{K}')$. We denote by $\theta$ here the conjugation in $\PG(n-1, \mathbb{K}')$ with respect to the field extension $\mathbb{K}/\mathbb{K}$. There exists an $(\frac{n}{2}-1)$-dimensional subspace $\beta$ of $\PG(n-1, \mathbb{K}')$ such that $\beta \cap \beta^\theta = \emptyset$. For every point $x \in \beta$, $L_x := xx^\theta \cap \PG(n-1, \mathbb{K})$ is a line of $\PG(n-1, \mathbb{K})$. The set $S = \{L_x \mid x \in \beta\}$ is a spread of $\PG(n-1, \mathbb{K})$, i.e. a set of lines of $\PG(n-1, \mathbb{K})$ partitioning the point-set of $\PG(n-1, \mathbb{K})$. Any spread of $\PG(n-1, \mathbb{K})$ which can be obtained in this way is called a regular spread. For a discussion of regular spreads in the finite case, see Hirschfeld [12, Chapter 4] and [13, Chapter 17].

Let $X$ be as defined in Proposition 4.7 and let $x$ be a point of $X$. The convex subspaces of $DQ^-(2n+1, \mathbb{K})$ containing the point $x$ define a projective space $L_x$ isomorphic to $\PG(n-1, \mathbb{K})$. The quads through $x$ containing at least two points of $X$ define a spread $S_x$ of $L_x$ by property (SDPS5).

**Proposition 4.8** For every point $x$ of $X$, the spread $S_x$ of $L_x$ is regular.

**Proof.** Let $\gamma$ be the generator of $Q^-(2n+1, \mathbb{K})$ corresponding to $x$. Then there exists a generator $\beta$ of $H$ such that $\gamma = \langle \beta, \beta^\theta \rangle \cap \PG(2n+1, \mathbb{K})$. The lines of the spread $S_x$ of $L_x$ correspond to the subspaces $\langle \eta, \eta^\theta \rangle \cap \PG(2n+1, \mathbb{K}) = \langle \eta, \beta^\theta \rangle \cap \PG(2n+1, \mathbb{K})$, where $\eta$ is some hyperplane of $\beta$. In this way, we obtain a regular spread in the dual projective space associated to $\gamma$. This proves the proposition.

The following proposition is precisely Theorem 1.2.
Proposition 4.9 Any two geometric SDPS-sets of $DQ^-(2n+1,\mathbb{K})$ are isomorphic.

Proof. Let $V$ be the subspace of $V(2n+2,\mathbb{K}')$ whose nonzero elements consist of all vectors $\bar{x}$ for which $<\bar{x},\bar{x}>\in\alpha$. For all vectors $\bar{x}$ and $\bar{y}$ of $V$, we define $H(\bar{x},\bar{y}) := B(\bar{x},\bar{y})$. Then $H(\cdot,\cdot)$ is a Hermitian form on $V$ and $H$ is the Hermitian variety of $\alpha$ associated to it. Let $\delta$ be an element of $\mathbb{K}'$ such that $\delta^0 \not\in \{\delta,-\delta\}$. [If $\text{char}(\mathbb{K}) = 2$, then $\delta$ is an arbitrary element of $\mathbb{K'} \setminus \mathbb{K}$. If $\text{char}(\mathbb{K}) \neq 2$, then for an arbitrary $\mu \in \mathbb{K'} \setminus \mathbb{K}$, $\delta$ can be chosen in the set $\{\mu,\mu+1\}$.] Now, we can always choose a $k \in \mathbb{K} \setminus \{0\}$ and vectors $\bar{f}_0, \bar{f}_i, (i \in \{1,\ldots,\frac{n}{2}\})$, $\bar{g}_i, (i \in \{1,\ldots,\frac{n}{2}\})$ in $V$ such that

- $\alpha = \langle \bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{\frac{n}{2}}, \bar{g}_1, \ldots, \bar{g}_{\frac{n}{2}} \rangle$,
- $H(\bar{f}_0, \bar{f}_0) = -k(\delta - \delta^0)^2$,
- $H(\bar{f}_0, \bar{f}_i) = H(\bar{f}_0, \bar{g}_i) = 0$ for all $i \in \{1,\ldots,\frac{n}{2}\}$,
- $H(\bar{f}_i, \bar{f}_j) = H(\bar{g}_i, \bar{g}_j) = 0$ for all $i,j \in \{1,\ldots,\frac{n}{2}\}$,
- $H(\bar{f}_i, \bar{g}_i) = k \cdot \frac{\delta^0 - \delta}{\delta^0 - \delta^0}$ for every $i \in \{1,\ldots,\frac{n}{2}\}$,
- $H(\bar{f}_i, \bar{g}_j) = 0$ for all $i,j \in \{1,\ldots,\frac{n}{2}\}$ with $i \neq j$.

[If $\beta_1$ and $\beta_2$ are two disjoint generators of $H$ and $p = \langle \beta_1, \beta_2 \rangle^\zeta$, where $\zeta$ is the Hermitian polarity of $\alpha$ associated to $H$, then we can choose $\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_{\frac{n}{2}}, \bar{g}_1, \ldots, \bar{g}_{\frac{n}{2}}$ in such a way that $p = \langle \bar{f}_0 \rangle$, $\beta_1 = \langle \bar{f}_1, \ldots, \bar{f}_{\frac{n}{2}} \rangle$ and $\beta_2 = \langle \bar{g}_1, \ldots, \bar{g}_{\frac{n}{2}} \rangle$.] Now, put

$$\bar{e}_0 = \frac{\bar{f}_0 - \bar{f}_0}{\delta^0 - \delta}, \quad \bar{e}_1 = \frac{\delta \bar{f}_0^0 - \delta^0 \bar{f}_0}{\delta^0 - \delta},$$

and

$$\bar{e}_{4i-2} = \frac{\delta \bar{f}_0 - \delta \bar{f}_0}{\delta^0 - \delta}, \quad \bar{e}_{4i} = \frac{\bar{f}_i - \bar{f}_i}{\delta^0 - \delta},$$

$$\bar{e}_{4i-1} = \frac{\delta \bar{g}_i - \delta \bar{g}_i}{\delta^0 - \delta}, \quad \bar{e}_{4i+1} = \frac{\delta \bar{g}_i^0 - \bar{g}_i}{\delta^0 - \delta},$$

for every $i \in \{1,\ldots,\frac{n}{2}\}$. Then $\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n+1} \in V(2n+2,\mathbb{K})$. Moreover, these $2n+2$ vectors are linearly independent since $\alpha \cap \alpha^0 = \emptyset$. Hence, $(\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n+1})$ is a reference system for PG$(2n+1,\mathbb{K})$. Suppose

$$q(\sum_{i=0}^{2n+1} X_i \bar{e}_i) = \sum_{0 \leq i \leq 2n+1} a_{ij} X_i X_j.$$
Let $i \in \{1, \ldots, n \}$. Since $\langle \hat{f}_i \rangle \in H$, $\langle \hat{f}_i \rangle$ and $\langle \hat{f}_1^\theta \rangle$ are collinear points on $Q^+(2n + 1, \mathbb{K})$. Hence, $\langle \hat{e}_{4i-2} \rangle, \langle \hat{e}_{4i} \rangle \in Q^-(2n + 1, \mathbb{K})$. In a similar way, one can prove that $\langle \hat{e}_{4i-1} \rangle, \langle \hat{e}_{4i+1} \rangle \in Q^-(2n + 1, \mathbb{K})$. We can conclude that $a_{ii} = 0$ for every $i \in \{2, \ldots, 2n + 1\}$.

Notice that since $\alpha$ is a generator of $Q^+(2n + 1, \mathbb{K}^\prime)$, $B(\bar{x}, \bar{y}) = H(\bar{x}, \bar{y}^\theta) = 0$ for all $\bar{x}, \bar{y} \in \{ \bar{f}_0, \bar{f}_1, \ldots, \bar{f}_2, \bar{g}_1, \ldots, \bar{g}_2 \}$.

We calculate

$$a_{01} = B(\bar{e}_0, \bar{e}_1)$$

$$= B(\bar{f}_0^\theta - \bar{f}_0, \delta \bar{f}_0^\theta - \delta^\theta \bar{f}_0)$$

$$= \frac{\delta \cdot B(\bar{f}_0^\theta, \bar{f}_0^\theta) - \delta^\theta \cdot B(\bar{f}_0^\theta, \bar{f}_0) - \delta \cdot B(\bar{f}_0, \bar{f}_0^\theta) + \delta^\theta \cdot B(\bar{f}_0, \bar{f}_0)}{(\delta^\theta - \delta)^2}.$$

Now, $B(\bar{f}_0, \bar{f}_0) = 0$, $B(\bar{f}_0^\theta, \bar{f}_0^\theta) = (B(\bar{f}_0, \bar{f}_0))^\theta = 0$ and $B(\bar{f}_0^\theta, \bar{f}_0) = B(\bar{f}_0, \bar{f}_0^\theta) = H(\bar{f}_0, \bar{f}_0) = -k(\delta - \delta^\theta)^2$. It follows that

$$a_{01} = k(\delta + \delta^\theta).$$

After some straightforward calculations, one finds in a similar way that

- $a_{ii} = a_{ii} = 0$ for all $i \in \{2, \ldots, 2n + 1\}$,
- $a_{2i,2i+1} = k$ for all $i \in \{1, \ldots, n\}$,
- $a_{j_1,j_2} = 0$ for all $j_1, j_2 \in \{2, \ldots, 2n + 1\}$ with $j_1 < j_2$ and $(j_1, j_2)$ not of the form $(2i, 2i + 1)$ for some $i \in \{1, \ldots, n\}$.

Now, since $\langle \hat{f}_0 \rangle = \langle \delta \bar{e}_0 - \bar{e}_1 \rangle \in \alpha$ and $\langle \hat{f}_0^\theta \rangle = \langle \delta^\theta \bar{e}_0 - \bar{e}_1 \rangle \in \alpha^\theta$ are points of $Q^+(2n + 1, \mathbb{K}^\prime)$, we have

$$\left\{ \begin{array}{ll}
a_{00} \cdot \delta^2 + k(\delta + \delta^\theta)(-\delta) + a_{11} & = 0, \\
a_{00} \cdot (\delta^\theta)^2 + k(\delta + \delta^\theta)(-\delta^\theta) + a_{11} & = 0.
\end{array} \right.$$}

Hence, $a_{00} = k$ and $a_{11} = k\delta^{\theta+1}$. So, with respect to the reference system $(\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_{2n+1})$ of $PG(2n + 1, \mathbb{K})$, $Q^-(2n + 1, \mathbb{K})$ has equation

$$X_0^2 + (\delta + \delta^\theta)X_0X_1 + \delta^{\theta+1}X_1^2 + X_2X_3 + \cdots + X_{2n}X_{2n+1} = 0.$$
\((\bar{e}_0^1, \bar{e}_1^1, \ldots, \bar{e}_{2n+1}^1)\) of \(\text{PG}(2n+1, \mathbb{K})\), \(Q^-(2n+1, \mathbb{K})\) has also equation \(X_0^2 + (\delta + \delta^\theta)X_0X_1 + \delta^{\theta + 1}X_1^2 + X_2X_3 + \cdots + X_{2n}X_{2n+1} = 0\). It is now clear that the linear map \(\sum_{i=0}^{2n+1} X_i \bar{e}_i \mapsto \sum_{i=0}^{2n+1} X_i e_i^1\) of \(V(2n+2, \mathbb{K}')\) induces an automorphism of \(\text{PG}(2n+1, \mathbb{K}')\) fixing \(\text{PG}(2n+1, \mathbb{K})\) and \(Q^-(2n+1, \mathbb{K})\) setwise and mapping \(\alpha = (\bar{f}_0^1, \bar{f}_1^1, \ldots, \bar{f}_2^1, \bar{g}_1^1, \ldots, \bar{g}_2^1)\) to \(\alpha^\dagger = (f_0^1, f_1^1, \ldots, f_2^1, g_1^1, \ldots, g_2^1)\). Hence, the geometric SDPS-sets of \(DQ^-(2n+1, \mathbb{K})\) associated to \(\alpha\) and \(\alpha^\dagger\) are isomorphic.

\section{5 The natural embedding of \(DQ^-(2n+1, \mathbb{K})\) into the half-spin geometry for \(Q^+(2n+1, \mathbb{K}')\)}

We will continue with the notation introduced in Section 3. For every \(\alpha \in \mathcal{M}\) and every generator \(\gamma\) of \(Q^-(2n+1, \mathbb{K})\), we define
\[
f_\alpha(\gamma) := M - \dim(\gamma' \cap \alpha),
\]
where
\[
M := \max\{\dim(\eta' \cap \alpha) \mid \eta\text{ is a generator of }Q^-(2n+1, \mathbb{K})\}.
\]

\textbf{Proposition 5.1} For every \(\alpha \in \mathcal{M}\), \(f_\alpha\) is a valuation of the dual polar space \(DQ^-(2n+1, \mathbb{K})\) associated to \(Q^-(2n+1, \mathbb{K})\).

\textbf{Proof.} By definition, the minimal value attained by \(f_\alpha\) is equal to 0. So, property (V1) is satisfied.

Let \(\beta\) be an arbitrary \((n-2)\)-dimensional subspace of \(Q^-(2n+1, \mathbb{K})\). Then there exists a unique generator \(\eta\) of \(Q^+(2n+1, \mathbb{K}')\) through \(\beta'\) for which \(\dim(\alpha \cap \eta) = \dim(\alpha \cap \beta') + 2\). Let \(\gamma\) be the unique subspace of \(\text{PG}(2n+1, \mathbb{K})\) such that \(\gamma' = \eta \cap \eta^\theta\) (see Lemma 3.1). Then \(\gamma \subseteq Q^-(2n+1, \mathbb{K})\) and \(\beta' \subseteq \gamma' \subseteq \eta\). By Lemma 3.2, \(\dim(\gamma') = n - 1\). So, \(\gamma\) is a generator of \(Q^-(2n+1, \mathbb{K})\) through \(\beta\). Since \(\beta' \subseteq \gamma' \subseteq \eta\) and \(\dim(\alpha \cap \eta) = \dim(\alpha \cap \beta') + 2\), \(\dim(\alpha \cap \gamma') = \dim(\alpha \cap \beta') + 1\). Conversely, suppose that \(\kappa\) is a generator of \(Q^-(2n+1, \mathbb{K})\) through \(\beta\) such that \(\dim(\alpha \cap \kappa') = \dim(\alpha \cap \beta') + 1\). Then \(\kappa'\) is necessarily contained in \(\eta\). Then \(\kappa' = \kappa^\theta \subseteq \eta^\theta\) and \(\kappa' \subseteq \eta \cap \eta^\theta = \gamma'\). Since \(\kappa'\) and \(\gamma'\) have the same dimension, we have \(\kappa = \gamma\). It follows that the line of \(DQ^-(2n+1, \mathbb{K})\) corresponding to \(\beta\) has a unique point with smallest \(f_\alpha\)-value, namely the point corresponding to \(\gamma\), and that all the remaining points of that line have value \(f_\alpha(\gamma) + 1\). This proves that property (V2) is satisfied.

Now, let \(\beta\) be an arbitrary generator of \(Q^-(2n+1, \mathbb{K})\). By Lemma 3.1, there exists a subspace \(\gamma\) of \(\text{PG}(2n+1, \mathbb{K})\) such that \(\gamma' = (\alpha \cap \beta', \alpha^\theta \cap \beta') \cap

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\( (\alpha \cap \beta', \alpha^\theta \cap \beta')^\theta = (\alpha \cap \beta', \alpha^\theta \cap \beta') \subseteq \beta' \). Let \( F_\beta \) denote the convex subspace of \( DQ^- (2n + 1, \mathbb{K}) \) corresponding to the subspace \( \gamma \) of \( Q^- (2n + 1, \mathbb{K}) \). Obviously, the point of \( DQ^- (2n + 1, \mathbb{K}) \) corresponding to \( \beta \) belongs to \( F_\beta \).

We will now prove that property (V3) is satisfied with respect to the convex subspace \( F_\beta \). Let \( \eta \) be a generator of \( Q^- (2n + 1, \mathbb{K}) \) through \( \gamma \). Then \( \eta' \) contains \( \gamma' = (\alpha \cap \beta', \alpha^\theta \cap \beta') \) and hence \( \dim(\eta' \cap \alpha) \geq \dim(\alpha \cap \beta') \), i.e. \( f_\alpha(\eta) \leq f_\alpha(\beta) \). Now, let \( \kappa \) be an arbitrary generator of \( Q^- (2n + 1, \mathbb{K}) \) such that \( f_\alpha(\kappa) = f_\alpha(\eta) - 1 \) and \( \dim(\eta \cap \kappa) = n - 2 \). So, \( \dim(\alpha \cap \kappa') = \dim(\alpha \cap \eta') + 1 \) and \( \dim(\kappa' \cap \eta') = n - 2 \). Let \( p \) be an arbitrary point of \( (\alpha \cap \kappa') \setminus (\alpha \cap \eta') \). Then \( \kappa' \cap \eta' \) is the set of points of \( \eta' \) collinear with \( p \) on \( Q^+ (2n + 1, \mathbb{K}') \). Since every point of \( \alpha \cap \eta' \) is collinear on \( Q^+ (2n + 1, \mathbb{K}') \) with \( p \in \alpha, \alpha \cap \eta' \subseteq \eta' \cap \kappa', \) i.e. \( \alpha \cap \eta' \subseteq \kappa' \). Hence, also \( \alpha^\theta \cap \eta' \subseteq \kappa' \).

Proposition 5.2 Suppose there exists a generator \( \beta \) of \( Q^- (2n + 1, \mathbb{K}) \) such that \( \beta' \subseteq \alpha \). Then \( f_\alpha \) is a classical valuation of \( DQ^- (2n + 1, \mathbb{K}) \), namely, for every generator \( \gamma \) of \( Q^- (2n + 1, \mathbb{K}) \), \( f_\alpha(\gamma) \) equals the distance \( d(\beta, \gamma) \) between \( \beta \) and \( \gamma \) in the dual polar space \( DQ^- (2n + 1, \mathbb{K}) \).

Proof. From \( \beta' \subseteq \alpha \), it follows \( \beta' = \beta^\theta \subseteq \alpha^\theta \) and hence \( \beta' = \alpha \cap \alpha^\theta \) (recall Lemma 3.2). Let \( \gamma \) be an arbitrary generator of \( Q^- (2n + 1, \mathbb{K}) \). Suppose \( \gamma' \) contains a point \( x \) of \( \alpha \setminus \beta' \). Since \( x, x^\theta \in \gamma' \subseteq Q^+ (2n + 1, \mathbb{K}') \), \( xx^\theta \subseteq Q^+ (2n + 1, \mathbb{K}') \) and \( \langle \alpha, \alpha^\theta \rangle \subseteq Q^+ (2n + 1, \mathbb{K}') \). This is impossible since \( \alpha \) and \( \alpha^\theta \) are generators of \( Q^+ (2n + 1, \mathbb{K}') \). Hence, \( \gamma' \cap \alpha \subseteq \beta' \), i.e. \( \gamma' \cap \alpha = \gamma' \cap \beta' \).

Lemma 5.3 Let \( x \) be a point of \( \alpha \cap \alpha^\theta \cap \text{PG}(2n + 1, \mathbb{K}) \), let \( \beta \) be a generator of \( Q^- (2n + 1, \mathbb{K}) \) not containing \( x \) and let \( \gamma \) be the unique generator of \( Q^- (2n + 1, \mathbb{K}) \) containing \( x \) intersecting \( \beta \) in a subspace of dimension \( n - 2 \). Then \( \dim(\gamma' \cap \alpha) = \dim(\beta' \cap \alpha) + 1 \).

Proof. Since \( x \in \gamma' \setminus \beta' \), \( \beta' \neq \gamma' \). So, \( \beta' \cap \gamma' \) is a hyperplane of both \( \beta' \) and \( \gamma' \) and \( \dim(\gamma' \cap \alpha) \leq \dim(\beta' \cap \gamma' \cap \alpha) + 1 \leq \dim(\beta' \cap \alpha) + 1 \). We will now prove that \( \dim(\beta' \cap \alpha) + 1 \leq \dim(\gamma' \cap \alpha) \). If \( x \in Q^- (2n + 1, \mathbb{K}) \) were collinear on \( Q^+ (2n + 1, \mathbb{K}') \) with every point of \( \beta' \), then \( x \) would also be collinear on \( Q^- (2n + 1, \mathbb{K}) \) with every point of \( \beta \), contradicting the fact that \( x \in Q^- (2n + 1, \mathbb{K}) \setminus \beta \) and \( \beta \) is a generator of \( Q^- (2n + 1, \mathbb{K}) \). Hence, the
points of \( \beta' \) collinear on \( Q^+(2n+1, K') \) with \( x \) form a hyperplane of \( \beta' \) which necessarily coincides with \( (\beta \cap \gamma)' \). Since every point of \( \beta' \cap \alpha \) is collinear on \( Q^+(2n+1, K') \) with \( x \in \alpha, \beta' \cap \alpha \subseteq (\beta \cap \gamma)' = \beta' \cap \gamma' \). Hence, \( \beta' \cap \alpha \subseteq \gamma' \cap \alpha \). Now, since \( x \in (\gamma' \cap \alpha) \setminus (\beta' \cap \alpha) \), we have \( \dim(\gamma' \cap \alpha) \geq \dim(\beta' \cap \alpha) + 1 \).

Suppose \( x \) is a point of \( \alpha \cap \alpha^G \cap PG(2n+1, K) \), where \( n \geq 3 \). The subspaces of \( Q^-(2n+1, K) \) (respectively \( Q^+(2n+1, K') \)) through \( x \) define a polar space \( Q^-(2n-1, K) \) (respectively \( Q^+(2n-1, K') \)). The maximal subspaces of \( Q^-(2n+1, K) \) through \( x \) form a max \( M \cong DQ^-(2n-1, K) \) of \( DQ^-(2n+1, K) \). Since \( \alpha \) is a maximal subspace of \( Q^+(2n+1, K') \), we can define a valuation \( f^M_\alpha \) of \( M \), similarly as we could define the valuation \( f^M_\alpha \) of \( DQ^- (2n+1, K) \) from the maximal subspace \( \alpha \) of \( Q^+(2n+1, K') \). From Lemma 5.3, we immediately obtain:

**Proposition 5.4** Let \( f^M_\alpha \) be as defined before this proposition. Then the valuation \( f_\alpha \) of \( DQ^-(2n+1, K) \) is the extension of the valuation \( f^M_\alpha \) of \( M \).

**Proposition 5.5** (i) If \( n = 2 \), then \( f_\alpha \) is a classical or ovoidal valuation of \( DQ^-(5, K) \).

(ii) If \( n = 3 \), then the valuation \( f_\alpha \) of \( DQ^-(7, K) \) is either a classical valuation or the extension of an ovoidal valuation of a quad of \( DQ^-(7, K) \).

**Proof.** (i) If \( n = 2 \), then \( \alpha \) is a generator of \( Q^+(5, K') \). Since \( \alpha \) and \( \alpha^G \) belong to different systems of generators of \( Q^+(5, K') \), \( \alpha \cap \alpha^G \) is either a line or the empty set. If \( \alpha \cap \alpha^G \) is a line, then \( f_\alpha \) is a classical valuation of \( DQ^-(5, K) \) by Lemma 3.1 and Proposition 5.2. Suppose therefore that \( \alpha \cap \alpha^G = \emptyset \). Then \( \dim(\beta' \cap \alpha) \leq 0 \) for every generator (= line) \( \beta \) of \( Q^-(5, K) \). It follows that \( f_\alpha \) can only attain the values 0 and 1. This implies that \( f_\alpha \) is an ovoidal valuation of \( DQ^-(5, K) \).

(ii) If \( n = 3 \), then since \( \alpha \) and \( \alpha^G \) belong to different systems of generators of \( Q^+(7, K') \), \( \dim(\alpha \cap \alpha^G) \in \{0, 2\} \). By Lemma 3.1, there exists a point \( x \in \alpha \cap \alpha^G \cap PG(2n+1, K) \). Claim (ii) follows from Claim (i) and Proposition 5.4. (Notice that extensions of classical valuations are again classical.)

**Proposition 5.6** The valuation \( f_\alpha \) is the possibly trivial extension of an SDPS-valuation of a convex subspace of \( DQ^-(2n+1, K) \).

**Proof.** Let \( DQ^+(2n+1, K') \) denote the dual polar space associated to \( Q^+(2n+1, K') \) and let \( d^+ (\cdot, \cdot) \) denote the distance function in \( DQ^+(2n+1, K') \).

By Proposition 5.5, the proposition holds if \( n \leq 3 \). So, suppose \( n \geq 4 \). Let \( U \) denote an arbitrary hex of \( DQ^-(2n+1, K) \) corresponding to an \( (n-4) \)-dimensional subspace \( \beta \) of \( Q^-(2n+1, K) \). The subspace \( \beta' \) of \( Q^+(2n+1, K') \)
corresponds to a convex subspace $F$ of diameter 4 of $DQ^+(2n + 1, \mathbb{K}')$. Let $\tilde{\alpha}$ denote the unique point of $F$ nearest to $\alpha$. For every generator $\gamma$ of $Q^-(2n + 1, \mathbb{K})$ through $\beta$, put

$$\tilde{f}_{\alpha}(\gamma) = \tilde{M} - \dim(\gamma' \cap \tilde{\alpha}),$$

where

$$\tilde{M} := \max\{\dim(\eta' \cap \tilde{\alpha}) \mid \eta \text{ is a generator of } Q^-(2n + 1, \mathbb{K}) \text{ through } \beta\}.$$ 

Then $\tilde{f}_{\alpha}$ is a valuation of $U$, which by Proposition 5.5 is either a classical valuation or the extension of an ovoidal valuation of a quad of $U$.

Now, for every generator $\gamma$ of $Q^-(2n + 1, \mathbb{K})$ through $\beta$, $n - 1 - \dim(\gamma' \cap \alpha)$ is equal to the distance $d^+(\gamma', \alpha)$ between the line $\gamma'$ of $DQ^+(2n + 1, \mathbb{K}')$ and the point $\alpha$ of $DQ^+(2n + 1, \mathbb{K}')$. Since $F$ is classical in $DQ^+(2n + 1, \mathbb{K}')$, $d^+(\gamma', \alpha) = d^+(\gamma', \tilde{\alpha}) + d^+(\tilde{\alpha}, \alpha)$ and hence $\dim(\gamma' \cap \alpha) = n - 1 - d^+(\gamma', \alpha) = n - 1 - d^+(\gamma', \tilde{\alpha}) - d^+(\tilde{\alpha}, \alpha) = \dim(\gamma' \cap \tilde{\alpha}) - d^+(\tilde{\alpha}, \alpha)$. So, $f_{\alpha}(\gamma) = M - \dim(\gamma' \cap \alpha) = M + d^+(\tilde{\alpha}, \alpha) - \dim(\gamma' \cap \tilde{\alpha}) = M + d^+(\tilde{\alpha}, \alpha) - M + \tilde{f}_{\alpha}(\gamma)$. It follows that $f_{\alpha}$ is the valuation of $U$ induced by $\tilde{f}_{\alpha}$. Since $U$ was arbitrary, every induced hex-valuation is either classical or the extension of an ovoidal valuation of a quad. By Proposition 2.4, it now follows that $f_{\alpha}$ is the possibly trivial extension of an SDPS-valuation of a convex subspace of $DQ^-(2n + 1, \mathbb{K})$. ■

**Definition.** Let $F_{\alpha}$ denote the convex subspace of $DQ^-(2n + 1, \mathbb{K})$ such that $f_{\alpha}$ is the extension of an SDPS-valuation of $F_{\alpha}$. Let $X_{\alpha}$ denote the SDPS-set of $F_{\alpha}$ corresponding to the SDPS-valuation of $F_{\alpha}$ giving rise to $f_{\alpha}$. The set $X_{\alpha}$ consists of those points of $DQ^-(2n + 1, \mathbb{K})$ whose $f_{\alpha}$-value is equal to 0, or equivalently, consists of those generators $\gamma$ of $Q^-(2n + 1, \mathbb{K})$ for which $\dim(\gamma' \cap \alpha)$ attains its maximal value $M$.

**Proposition 5.7** $F_{\alpha}$ is the convex subspace of $DQ^-(2n + 1, \mathbb{K})$ corresponding to the subspace $(\alpha \cap \alpha^\theta) \cap PG(2n + 1, \mathbb{K})$ of $Q^-(2n + 1, \mathbb{K})$ and $X_{\alpha}$ is a geometric SDPS-set in $F_{\alpha}$.

**Proof.** (i) Suppose first that $\alpha \cap \alpha^\theta = \emptyset$. Then $n$ is even by Lemma 3.2. Recall that $M$ is the maximal value of $\dim(\gamma' \cap \alpha)$, where $\gamma$ ranges over all generators of $Q^-(2n + 1, \mathbb{K})$. Let $H$ denote the set of points $x$ of $\alpha$ which are collinear on $Q^+(2n + 1, \mathbb{K}')$ with $x^\theta$. Then by Proposition 4.6, $H$ is a nonsingular $\theta$-Hermitian variety of Witt index $\frac{n}{2}$ in $\alpha$. The set $X$ of generators of $Q^-(2n + 1, \mathbb{K})$ of the form $\langle \beta, \beta^\theta \rangle \cap PG(2n + 1, \mathbb{K})$ where $\beta$ is some generator of $H$ is a (geometric) SDPS-set of $DQ^-(2n + 1, \mathbb{K})$. 

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If $\gamma$ is a generator of $Q^{-}(2n+1, \mathbb{K})$, then $\gamma' \cap \alpha$ and $(\gamma' \cap \alpha)^{0} = \gamma' \cap \alpha^{0}$ are disjoint subspaces of $\gamma'$. Since $\dim(\gamma') = n - 1$, we necessarily have $\dim(\gamma' \cap \alpha) \leq \frac{n}{2} - 1$. Hence, $M \leq \frac{n}{2} - 1$.

If $\beta$ is a generator of $H$, then $\gamma = \langle \beta, \beta^{0} \rangle \cap \text{PG}(2n+1, \mathbb{K})$ is a generator of $Q^{-}(2n+1, \mathbb{K})$. Moreover, $\gamma' \cap \alpha = \langle \beta, \beta^{0} \rangle \cap \alpha = \beta$ has dimension $\frac{n}{2} - 1$.

So, we can conclude that $M = \frac{n}{2} - 1$. It is clear from the above that the generators $\gamma$ of $Q^{-}(2n+1, \mathbb{K})$ for which $\dim(\gamma' \cap \alpha) = \frac{n}{2} - 1$ are precisely those generators of $Q^{-}(2n+1, \mathbb{K})$ which are of the form $\langle \beta, \beta^{0} \rangle \cap \text{PG}(2n+1, \mathbb{K})$ for some generator $\beta$ of $H$. So, $X_{\alpha} = X$ is a geometric SDPS-set of $DQ^{-}(2n+1, \mathbb{K})$. Since $DQ^{-}(2n+1, \mathbb{K})$ is the convex subspace of $DQ^{-}(2n+1, \mathbb{K})$ corresponding to the subspace $\emptyset = \alpha \cap \alpha^{0} \cap \text{PG}(2n+1, \mathbb{K})$ of $Q^{-}(2n+1, \mathbb{K})$, we have proved our claim.

(ii) Suppose $\beta := \alpha \cap \alpha^{0} \cap \text{PG}(2n+1, \mathbb{K}) \neq \emptyset$. Let $F$ denote the convex subspace of $DQ^{-}(2n+1, \mathbb{K})$ corresponding to $\beta$. By successive application of Proposition 5.4, we see that $f_{\alpha}$ is the extension of a valuation $f^{F}_{\alpha}$ of $F$. So, $X_{\alpha}$ must be a set of points of $F$. Now, taking the quotient polar spaces $P$ and $P'$ obtained by considering all subspaces of $Q^{-}(2n+1, \mathbb{K})$ and $Q^{+}(2n+1, \mathbb{K}')$ through $\beta$ and $\beta'$, respectively, and applying (i), we see that $X_{\alpha}$ must be a geometric SDPS-set in $F$. ■

Now, put

$$K := \frac{n - 1 - \dim(\alpha \cap \alpha^{0})}{2}.$$ 

Then $K \in \mathbb{N}$ by Lemma 3.2. More precisely, we have $0 \leq K \leq \lfloor \frac{n}{2} \rfloor$. By Proposition 5.7, the diameter $\text{diam}(F_{\alpha})$ of $F_{\alpha}$ is equal to $(n - 1) - \dim(\alpha \cap \alpha^{0}) = 2K$. So, the maximal value of an SDPS-valuation of $F_{\alpha}$ is equal to $K$. It follows that the maximal value of $f_{\alpha}$ is equal to

$$K + \text{diam}(DQ^{-}(2n+1, \mathbb{K})) - \text{diam}(F_{\alpha}) = n - K.$$ 

In the following proposition, we determine the precise value of the parameter $M = \max\{\dim(\eta' \cap \alpha) \mid \eta \text{ is a generator of } Q^{-}(2n+1, \mathbb{K})\}$.

**Proposition 5.8** We have $M = n - K - 1$.

**Proof.** We have $\dim(\alpha \cap \alpha^{0}) = n - (2K + 1)$. Let $\pi$ be the $(n - 2K - 1)$-dimensional subspace of $\text{PG}(2n+1, \mathbb{K})$ such that $\pi' = \alpha \cap \alpha^{0}$. Taking the quotient of $Q^{-}(2n+1, \mathbb{K})$ and $Q^{+}(2n+1, \mathbb{K}')$ over the respective subspaces $\pi$ and $\pi' = \alpha \cap \alpha^{0}$, we obtain polar spaces isomorphic to $Q^{-}(4K+1, \mathbb{K})$ and $Q^{+}(4K+1, \mathbb{K}')$. By successive application of Lemma 5.3, we see that $M = \dim(\pi') + 1 + \text{dimension of a generator of } H(2K, \mathbb{K}', 0) = n - K - 1$. (Recall
that by the proof of Proposition 5.7, \( M \) equals the dimension of a generator of \( H \) if \( \alpha \cap \alpha^\phi = \emptyset \).

\[ \Box \]

**Corollary 5.9** There exists a generator \( \eta \) of \( Q^-(2n+1, \mathbb{K}) \) such that \( \eta' \cap \alpha = \emptyset \).

**Proof.** For every generator \( \eta \) of \( Q^-(2n+1, \mathbb{K}) \), we have \( f_\alpha(\eta) = M - \dim(\eta' \cap \alpha) = n - K - 1 - \dim(\eta' \cap \alpha) \). The claim now follows from the fact that the maximal value of \( f_\alpha(\eta) \) is equal to \( n - K \).

\[ \Box \]

Now, let \( \epsilon \in \{+, -\} \) such that \( \alpha \in \mathcal{M}^\epsilon \). Recall that \( HS^\epsilon(2n+1, \mathbb{K}') \) denotes the half-spin geometry for \( Q^+(2n+1, \mathbb{K}') \) defined on the set \( \mathcal{M}^\epsilon \). Let \( d_\epsilon(\cdot, \cdot) \) denote the distance function in \( HS^\epsilon(2n+1, \mathbb{K}') \). For any two elements \( \alpha_1, \alpha_2 \in \mathcal{M}^\epsilon \), we have \( d_\epsilon(\alpha_1, \alpha_2) = \frac{n - \dim(\alpha_1 \cap \alpha_2)}{2} \). The diameter of \( HS^\epsilon(2n+1, \mathbb{K}') \) is equal to \( \lceil \frac{n+1}{2} \rceil \).

For every generator \( \gamma \) of \( Q^-(2n+1, \mathbb{K}) \), let \( \gamma^\phi \) denote the unique element of \( \mathcal{M}^\epsilon \) through \( \gamma' \). Then \( \phi \) defines a full embedding of \( \Delta = DQ^-(2n+1, \mathbb{K}) \) into \( HS^\epsilon(2n+1, \mathbb{K}') \) (the natural embedding of \( DQ^-(2n+1, \mathbb{K}) \) into \( HS^\epsilon(2n+1, \mathbb{K}') \)). Since \( \gamma^\phi \) and \( \alpha \) belong to the same system of generators of \( Q^+(2n+1, \mathbb{K}') \), \( n - \dim(\alpha \cap \gamma^\phi) \) is even. Obviously, \( \dim(\alpha \cap \gamma^\phi) - \dim(\alpha \cap \gamma') \in \{0, 1\} \). Hence, \( n - \dim(\alpha \cap \gamma^\phi) = 2 \cdot \lceil \frac{n - \dim(\alpha \cap \gamma')}{2} \rceil \), i.e.

\[ d_\epsilon(\alpha, \gamma^\phi) = \frac{n - \dim(\alpha \cap \gamma')}{2}. \]

Since the maximal value of \( \dim(\alpha \cap \eta') \), where \( \eta \) ranges over all generators of \( Q^-(2n+1, \mathbb{K}) \), is equal to \( n - K - 1 \), we have

\[ d_\epsilon(\alpha, \Delta^\phi) = \lceil \frac{K + 1}{2} \rceil. \]

Now, for every generator \( \gamma \) of \( Q^-(2n+1, \mathbb{K}) \),

\[ d_\epsilon(\alpha, \gamma^\phi) = \frac{n - \dim(\alpha \cap \gamma')}{2} \]

\[ = \frac{n - M + M - \dim(\alpha \cap \gamma')}{2} \]

\[ = \frac{K + 1 + f_\alpha(\gamma)}{2} \]

\[ = \frac{K + 1 + d_\Delta(\gamma, X_\alpha)}{2}. \]

This proves Theorem 1.3.
Remark. Since the maximal value of \( f_\alpha(\gamma) = d_\Delta(\gamma, X_\alpha) \) is equal to \( n - K \), the maximal value of \( d_\epsilon(\alpha, \gamma^\phi) \) is equal to \( \left\lfloor \frac{n+1}{2} \right\rfloor \) which is precisely the diameter of \( HS^s(2n + 1, \mathbb{K}') \).

We will now prove Theorem 1.4. We will need the following lemma, which is easy to prove, see e.g. [9, Lemma 2.5].

\textbf{Lemma 5.10} (i) If \( n \) is odd, then the set of elements of \( M^\epsilon \) meeting a given element of \( M^\epsilon \) is a hyperplane of \( HS^s(2n + 1, \mathbb{K}') \).

(ii) If \( n \) is even, then the set of elements of \( M^\epsilon \) meeting a given element of \( M^- \) is a hyperplane of \( HS^s(2n + 1, \mathbb{K}') \).

Now, suppose that \( n \) is even, that \( \alpha \in M^\epsilon \) and that \( \alpha \cap \alpha^\theta = \emptyset \). Then by Lemma 5.10 (ii) and Lemma 3.2, the set of elements of \( M^\epsilon \) meeting \( \alpha^\theta \) defines a hyperplane \( H_\alpha \) of \( HS^s(2n + 1, \mathbb{K}') \).

Let \( \gamma \) be an arbitrary generator of \( Q^-\left(2n + 1, \mathbb{K}\right) \). Then \( \gamma^\phi \in H_\alpha \) if and only if \( \dim(\alpha^\theta \cap \gamma^\phi) \geq 0 \). Now, \( \dim(\alpha^\theta \cap \gamma^\phi) - \dim(\alpha^\theta \cap \gamma) \in \{0, 1\} \) and \( \dim(\alpha^\theta \cap \gamma^\phi) \) is odd since \( \alpha^\theta \) and \( \gamma^\phi \) belong to different systems of generators. It follows that \( \gamma^\phi \in H_\alpha \) if and only if \( \dim(\alpha^\theta \cap \gamma') \geq 0 \).

\[
\dim(\alpha^\theta \cap \gamma') = \dim(\alpha \cap \gamma')
= M - f_\alpha(\gamma)
= n - K - 1 - f_\alpha(\gamma).
\]

So, \( \gamma^\phi \in H_\alpha \) if and only if \( f_\alpha(\gamma) \leq n - K - 1 \). Now, the maximal value of the valuation \( f_\alpha \) is equal to \( n - K \). So, \( \gamma^\phi \in H_\alpha \) if and only if \( \gamma \) belongs to the hyperplane of \( DQ^-\left(2n + 1, \mathbb{K}\right) \) associated to the SDPS-set \( X_\alpha \). Now, let \( e \) denote the spin embedding of \( HS^s(2n + 1, \mathbb{K}') \). Then by the main result of Shult [16] (see also Corollary 1.3 of [8] for an alternative proof) every hyperplane of \( HS^s(2n + 1, \mathbb{K}') \) arises from \( e \). In particular, \( H_\alpha \) arises from \( e \). Now, the map \( e \circ \phi \) defines a full embedding \( e' \) of \( DQ^-\left(2n + 1, \mathbb{K}\right) \) which is isomorphic to the spin embedding of \( DQ^-\left(2n + 1, \mathbb{K}\right) \). Since \( H_\alpha \) arises from \( e \), the hyperplane of \( DQ^-\left(2n + 1, \mathbb{K}\right) \) associated to the SDPS-set \( X_\alpha \) arises from \( e' \). This proves Theorem 1.4.

6 Appendix: An alternative construction for the unique geometric SDPS-set of \( DQ^-\left(2n + 1, \mathbb{K}\right) \)

In De Bruyn and Vandecasteele [11] a construction was given to obtain SDPS-sets of the dual polar space \( DQ^-\left(2n + 1, q\right) \). We recall this construction.
Consider the finite field $\mathbb{F}_{q^2}$ with $q^2$ elements and let $\mathbb{F}_q$ denote the subfield of order $q$ of $\mathbb{F}_{q^2}$. Let $\delta$ denote an arbitrary element of $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Consider the following bijection $\phi$ between the vector spaces $\mathbb{F}_q^{4n+2}$ and $\mathbb{F}_{q^2}^{2n+1}$:

$$\phi(X_1, X_2, \ldots, X_{4n+2}) = (X_1 + \delta X_2, \ldots, X_{4n+1} + \delta X_{4n+2}).$$

Let $\langle \cdot, \cdot \rangle$ be a nondegenerate Hermitian form of $\mathbb{F}_{q^2}^{2n+1}$. For every $\bar{x} \in \mathbb{F}_{q^2}^{2n+1}$, we define $h(\bar{x}) := \langle \bar{x}, \bar{x} \rangle$ and for every $\bar{x} \in \mathbb{F}_q^{4n+2}$, we define $q(\bar{x}) := \langle \phi(\bar{x}), \phi(\bar{x}) \rangle$.

The equation $h(\bar{x}) = 0$, respectively $q(\bar{x}) = 0$, defines a nonsingular Hermitian variety $H(2n, q^2)$ in $\text{PG}(2n, q^2)$, respectively a nonsingular elliptic quadric $Q^-(4n+1, q)$ in $\text{PG}(4n+1, q)$. With every generator of $H(2n, q^2)$, there corresponds (via the map $\phi^{-1}$) a generator of $Q^-(4n+1, q)$. The set of generators of $Q^-(4n+1, q)$ which arise in this way is an SDPS-set of $DQ^-(4n+1, q)$.

We will now show that the SDPS-sets which arise in this way are geometric. In order to facilitate the proof, we will give a slightly different but equivalent construction (which is presented here for possibly infinite fields).

Let $\mathbb{K}, \mathbb{K}', \theta, n, V(2n+2, \mathbb{K}), V(2n+2, \mathbb{K}')$, $\text{PG}(2n+1, \mathbb{K})$ and $\text{PG}(2n+1, \mathbb{K}')$ be as in Section 3 and suppose that $n$ is even. Let $V$ be an $(n+1)$-dimensional subspace of $V(2n+2, \mathbb{K}')$ such that $V^\theta \cap V = \{\delta\}$. Let $\{\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n\}$ be a basis of $V$ and let $\alpha$ be the subspace of $\text{PG}(2n+1, \mathbb{K}')$ corresponding to $V$. For every $i \in \{0, \ldots, n\}$, we define

$$\bar{f}_i := \bar{e}_i + \bar{e}_i^\theta,$$

$$\bar{g}_i := \delta \cdot \bar{e}_i + \delta^\theta \cdot \bar{e}_i^\theta,$$

where $\delta$ is some given element of $\mathbb{K}' \setminus \mathbb{K}$. Then $\{\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_n, \bar{g}_0, \bar{g}_1, \ldots, \bar{g}_n\}$ is a basis of $V(2n+2, \mathbb{K})$. Define the following bijection $\phi$ between $V(2n+2, \mathbb{K})$ and $V$:

$$\phi\left( \sum_{i=0}^{n} (X_i \bar{f}_i + Y_i \bar{g}_i) \right) := \sum_{i=0}^{n} (X_i + \delta Y_i) \bar{e}_i.$$

The following claim is obvious:

**Claim I**: For every vector $\bar{x} \neq \bar{\delta}$ of $V(2n+2, \mathbb{K})$, the point $\langle \bar{x} \rangle$ of $\text{PG}(2n+1, \mathbb{K}')$ is contained on the line connecting the points $\langle \phi(\bar{x}) \rangle \in \alpha$ and $\langle \phi(\bar{x})^\theta \rangle \in \alpha^\theta$.

Now, let $\langle \cdot, \cdot \rangle$ be a nondegenerate $\theta$-Hermitian form of $V$ of maximal Witt index $\frac{n}{2}$. For every vector $\bar{x}$ of $V(2n+2, \mathbb{K})$, we define

$$q(\bar{x}) := \langle \phi(\bar{x}), \phi(\bar{x}) \rangle.$$
Claim II: \( q \) is a nondegenerate quadratic form of Witt index \( n \) of \( V(2n + 2, K) \).

Proof. For every \( \bar{x} \in V(2n + 2, K) \) and every \( k \in K \), we have \( q(k \bar{x}) = k^2 q(\bar{x}) \). Now, for all \( \bar{x}_1, \bar{x}_2 \in V(2n + 2, K) \), we put \( B(\bar{x}_1, \bar{x}_2) := q(\bar{x}_1) - q(\bar{x}_2) = \langle \phi(\bar{x}_1), \phi(\bar{x}_2) \rangle \). Obviously, \( B \) is a symmetric \( K \)-bilinear form on \( V(2n + 2, K) \). We prove that \( B \) is nondegenerate. If \( B \) were degenerate, then there exists an \( \bar{x}^* \in V(2n + 2, K) \setminus \{ 0 \} \) such that \( \kappa(y) := \langle \phi(\bar{x}^*), y \rangle + \langle y, \phi(\bar{x}^*) \rangle = 0 \) for all \( y \in V \). From \( \kappa(y) = \kappa(\delta y) = 0 \), it then follows that \( \langle \phi(\bar{x}^*), y \rangle = 0 \) for all \( y \in V \). This contradicts the fact that \( \langle \cdot, \cdot \rangle \) is nondegenerate.

If \( U \) is a subspace of vector dimension \( \frac{n}{2} \) of \( V \) which is totally isotropic with respect to \( \langle \cdot, \cdot \rangle \), then \( q(\bar{x}) = 0 \) for every \( \bar{x} \in \phi^{-1}(U) \). Hence, the Witt index of \( q \) is at least \( n \). Suppose the Witt index of \( q \) is bigger than \( n \). Then there exists a vector \( \bar{x}^* \in V(2n + 2, K) \setminus \{ 0 \} \) not belonging to \( \phi^{-1}(U) \) such that \( q(\bar{x}) = 0 \) for every vector \( \bar{x} \) belonging to the subspace of \( V(2n + 2, K) \) generated by \( \bar{x}^* \) and \( \phi^{-1}(U) \). This implies that \( \kappa(y) := \langle \phi(\bar{x}^*), y \rangle + \langle y, \phi(\bar{x}^*) \rangle \) for any \( y \in U \). From \( \kappa(y) = \kappa(\delta y) = 0 \), it then follows that \( \langle \phi(\bar{x}^*), y \rangle = 0 \) for any \( y \in U \). Since also \( \langle \phi(\bar{x}^*), \phi(\bar{x}^*) \rangle = q(\bar{x}^*) = 0 \) and \( \phi(\bar{x}^*) \notin U \), this contradicts the fact that \( U \) is a maximal totally isotropic subspace of \( V \).

So, \( q \) is a nondegenerate quadratic form of Witt index \( n \) in \( V(2n + 2, K) \).\( \blacksquare \)

So, with \( q \) there is associated a nonsingular quadric \( Q \) of Witt index \( n \) in \( PG(2n + 1, K) \) and a quadric \( \tilde{Q} \) in \( PG(2n + 1, K') \). Since the bilinear form associated to \( q \) is nondegenerate, \( \tilde{Q} \) is a nonsingular quadric of Witt index \( n' \in \{ n, n + 1 \} \) in \( PG(2n + 1, K') \). Let \( H \) be the \( \theta \)-Hermitian variety of \( \alpha \) associated to \( \langle \cdot, \cdot \rangle \) and let \( \zeta \) denote the Hermitian polarity of \( \alpha \) associated to \( \langle \cdot, \cdot \rangle \). We will prove that \( \alpha \subseteq \tilde{Q} \). Let \( p = \langle \tilde{y} \rangle \) be an arbitrary point of \( \alpha \).

(a) Suppose first that \( p \in \tilde{H} \). By Claim I, for every point \( \langle \bar{x} \rangle \in pp^\theta \cap PG(2n + 1, K), \phi(\bar{x}) \) is a multiple of \( \tilde{y} \). Hence, the line \( pp^\theta \cap PG(2n + 1, K) \) of \( PG(2n + 1, K) \) is completely contained in \( Q^- (2n + 1, K) \). So, \( pp^\theta \subseteq \tilde{Q} \). In particular, \( p \in \tilde{Q} \).

(b) Suppose \( p \in \alpha \setminus \tilde{H} \). Clearly, there exists a point \( r \in \tilde{H} \setminus p^\perp \). For such a point \( r, rp \cap \tilde{H} \) is a Baer subline of \( rp \). Since each of the \( [K] + 1 \geq 3 \) points of \( rp \cap \tilde{H} \) are contained in \( \tilde{Q} \), the whole line \( rp \) is contained in \( \tilde{Q} \). In particular, \( p \in \tilde{Q} \).

Since \( \tilde{Q} \) contains subspaces of projective dimension \( n \), the Witt index of \( \tilde{Q} \) must be equal to \( n + 1 \). In the sequel, we will denote \( Q \) by \( Q^- (2n + 1, K) \) and \( \tilde{Q} \) by \( Q^+ (2n + 1, K') \). Now, let \( H \) denote the set of all points \( p \) of \( \alpha \) which are collinear on \( Q^+ (2n + 1, K') \) with \( p^\theta \). Clearly, \( p \in H \) if and only
if every point of \( p\theta \cap \text{PG}(2n + 1, \mathbb{K}) \) belongs to \( Q^-(2n + 1, \mathbb{K}) \). Now, a point \( \langle \bar{x} \rangle \in p\theta \cap \text{PG}(2n + 1, \mathbb{K}) \) belongs to \( Q^-(2n + 1, \mathbb{K}) \) if and only if \( p = \langle \phi(\bar{x}) \rangle \in \tilde{H} \). It follows that \( H = \tilde{H} \). Hence, the geometric SDPS-set of \( DQ^- (2n + 1, \mathbb{K}) \) associated to \( \alpha \) coincides with the set of all generators \( \langle U \rangle \) of \( Q^-(2n + 1, \mathbb{K}) \) for which \( \phi(U) \) is a maximal totally isotropic subspace of \( V \) with respect to the Hermitian form \( \langle \cdot, \cdot \rangle \). This is precisely what we needed to prove.

**Remark.** Although both constructions give rise to isomorphic SDPS-sets, there is an important difference between them. The construction described in this paper allows to obtain many (geometric) SDPS-sets in a given dual polar space isomorphic to \( DQ^- (2n + 1, \mathbb{K}) \). The other construction allows to obtain an SDPS-set in some dual polar space isomorphic to \( DQ^- (2n + 1, \mathbb{K}) \).

### References


