

Exact calculation of matrix entries in space-time Galerkin time domain boundary element method for the scalar wave equation

^{1st} Alessandro Zuccotti

Universiteit Gent

Gent, Belgium

alessandro.zuccotti@ugent.be

^{2nd} Kristof Cools

Universiteit Gent

Gent, Belgium

kristof.cools@ugent.be

Abstract—The contribution of this work is to propose an exact calculation for the interaction elements appearing in space-time time domain boundary element methods based on a dimensionality reduction procedure. Similar methods have been applied in the frequency domain, while in the time domain only numerical or semi-analytical approaches have been proposed. The calculation presented here replaces a four dimensional interaction integral with a sum of lower dimensional integrals of a new effective Green function that can be evaluated in closed form. In this paper the method proposed is applied to scalar basis and testing function on flat supports but it can be adapted to the electromagnetic case. The results can be used as a benchmark to evaluate the precision and stability of existing numerical methods. It allows to distinguish instabilities inherent to the method and those that appear as a consequence of limited quadrature precision.

I. INTRODUCTION

Time domain boundary integral equations allow modeling electromagnetic and acoustic scattering by systems comprising generic geometries and a wide range of materials. A general method to numerically solve this integral equation consists in using the marching-on-in-time (MoT) algorithm. A well known problem of the MoT algorithm is the late time instability, which can be caused, among other things, by the finite precision of the space-time interaction matrix elements. Indeed, in the Galerkin discretization, the interaction elements are of the form

$$I_{mn} = \int_{\Gamma} \int_{\Gamma'} f_m(r) t_n(r') \frac{\delta(t - \frac{R}{c})}{4\pi R} dS dS' * p(t), \quad (1)$$

where $f_m(r)$ and $t_n(r')$ are the spatial basis and testing function, Γ and Γ' their supports, $R = |r - r'|$, $p(t)$ is a function depending on the type of temporal discretization and $*$ is the convolution operator. The accurate computation of the interaction elements in both time and frequency domain has been the focus of various previous works. In the frequency domain the presence of singular kernels is the main obstacle for the accurate computation of the interaction elements. The general methods to deal with this are based on either the singularity cancellation or the singularity subtraction idea.

Another fully numerical approach was developed in [1] and [2] resulting in an analytical integrand that can be evaluated efficiently with the Gaussian quadrature method. In [3] Lenoir and Salles propose a method to reduce the dimensionality of the integration of homogenous Green functions that allows to compute exactly the interaction integral. A similar idea that leads to completely analytical evaluation is applied in the method of moments by Tihon and Craeye [4].

In time domain the convolution with the temporal basis function gives interaction integrals that cannot be accurately computed with the standard Gaussian method within reasonable computational costs, even when the singularity is out of the domain of integration. In [5], [6] more efficient and accurate numerical schemes were developed for both singular and nonsingular integrals. In [7] and [8] a closed form for the evaluation of the single surface integral is shown. A semi-analytical method for the double surface integral have been presented in [9], [10] and works by evaluating exactly three out of four integrals and computing numerically the remaining integral. In [11] the same method is used to get an efficient and stable MoT scheme for the EFIE, MFIE and CFIE.

In this work we propose a novel approach that allows to reduce the interaction integrals to a sum of lower dimensionality integrals that can be evaluated in closed form. We use the idea of the dimensionality reduction procedure for homogenous Green functions from [3] and we extend it to the time domain. We write the Green function in the frequency domain, we expand it in its power series (w.r.t. the interaction distance R) and we apply the dimensionality reduction procedure term by term. This is possible because each term in the series is homogeneous. The resulting series converges and gives a new effective Green function that can be evaluated exactly in a wide range of cases. In this work we restrict to the case of scalar and constant basis and testing function for its much simpler resulting formula, but the same idea will be used to build an analogous result for the electromagnetic case. Indeed, this method can be extended to polynomial basis and testing function of any degree but resulting in a higher number of

lower dimensional integrals to compute. The final goal of this method is to obtain a stable MoT scheme for those situations where the accuracy of the matrix elements has the highest impact on the late time stability of the solution, for example in case where the geometry supports many resonant modes or contains cavities leading to high Q-factors. The results for a pulse temporal basis function are presented and compared with a Gaussian quadrature method to test the validity of the formula.

II. DIMENSIONAL REDUCTION FOR HOMOGENEOUS GREEN FUNCTION

In this section we revisit the workings of the dimensional reduction procedure for a homogeneous integrand of the form

$$I = \int_{\Gamma} \int_{\Gamma'} f(r) t(r') H(R(r, r')) dS dS', \quad (2)$$

with $f(r)$ and $t(r')$ scalar functions which are constant within their supports, $R(r, r') = |r - r'|$, $H(R)$ an homogeneous function, and Γ, Γ' are assumed to be flat triangles. As in [3], by using the homogeneity property of the Green function the previous integral can be recast into

$$I = \frac{1}{d + d' + \alpha} \left(\sum_{i=1}^3 a_i \int_{\Gamma} \int_{\partial\Gamma'_i} f(r) t(r') H(R(r, r')) dS dl'_i + \sum_{j=1}^3 a'_j \int_{\partial\Gamma_j} \int_{\Gamma'} f(r) t(r') H(R(r, r')) dl_j dS' \right), \quad (3)$$

which is a sum of integrals with lower dimensionality of the same Green function. The domains $\partial\Gamma_i, \partial\Gamma'_j$ are the edges of Γ, Γ' , α is the degree of homogeneity of the Green function and $d, d' = 2$ are the dimensions of Γ, Γ' . The coefficients a_i, a'_j are determined upon choosing a point in the intersection of the planes defined by the two triangles. Such a point is characterized by parameter values satisfying

$$r(\lambda) = r'(\lambda'). \quad (4)$$

where $r(\lambda), r'(\lambda')$ is the parameterization of the triangles (for example λ, λ' can be the barycentric coordinates). Such system always has a line of solutions, except for the case of parallel triangles. In this case we use a different method that leads again to a reduced dimensionality. We do not show here such method since the general case where parallel triangles and edges can occur is beyond the scope of this contribution. An important property of this procedure is that the Green function in the right hand side of (3) is the same as that appearing in the original integral. We can reiterate this procedure in order to further reduce the dimensionality of the integrals. The 4-dimensional triangle-triangle integral is replaced by a sum of 2-dimensional integrals of the type vertex-triangle or edge-edge that can be evaluated exactly by a mild generalization of the techniques introduced in the seminal paper [12].

This calculation can be applied also to polynomial testing and

basis function by a recursion on both the degree of the trial/test functions and the dimension of the integration domain.

III. DIMENSIONAL REDUCTION IN TIME DOMAIN

In the MoT scheme for time domain BIEs, the Galerkin discretization produces interaction integrals of the type

$$I_{mn}(t) = \int_{\Gamma} \int_{\Gamma'} f_m(r) t_n(r') \frac{\delta(t - \frac{R}{c})}{4\pi R} dS dS' * p(t), \quad (5)$$

Again we assume Γ and Γ' to be planar triangles, $f(r)$ and $t(r')$ scalar constant functions, $R = |r - r'|$, $p(t)$ is the temporal basis function that is assumed to be piecewise polynomial and depends on the discretization scheme, and $*$ denotes temporal convolution. We propose a method that allows us to treat this integral by means of the same dimensionality reduction mentioned above. We first apply the Fourier transform obtaining

$$\tilde{I}_{mn}(\omega) = \int_{\Gamma} \int_{\Gamma'} f_m(r) t_n(r') \frac{e^{-i\omega \frac{R}{c}}}{4\pi R} \tilde{p}(\omega) dS dS'. \quad (6)$$

and then we expand the exponential in Taylor series around $R = 0$:

$$\tilde{I}_{mn}(\omega) = \tilde{p}(\omega) \int_{\Gamma} \int_{\Gamma'} \frac{f_m(r) t_n(r')}{4\pi R} \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\frac{-i\omega R}{c} \right)^n dS dS' \quad (7)$$

Each of the terms is of homogeneous type (2) and is amenable to the dimensional reduction procedure. Term n in the summation has degree of homogeneity $n - 1$ so we obtain

$$\tilde{I}_{mn}(\omega) = \tilde{p}(\omega) \sum_{i=1}^3 \left(a_i \int_{\partial\Gamma_i} \int_{\Gamma'} \frac{1}{4\pi R} \sum_{n=0}^{+\infty} \frac{1}{(n+3)n!} \left(\frac{-i\omega R}{c} \right)^n dl_i dS' + a'_i \int_{\Gamma} \int_{\partial\Gamma'_i} \frac{1}{4\pi R} \sum_{n=0}^{+\infty} \frac{1}{(n+3)n!} \left(\frac{-i\omega R}{c} \right)^n dS dl'_i \right), \quad (8)$$

where the factor $\frac{1}{n+3}$ corresponds to the coefficient in front of (3), which is given by the inverse of the sum of the dimension of the domain of integration and the degrees of homogeneity of the Green function. The series that now appears in the integrand is convergent and it gives a new effective green function of the reduced integrals. In our case we want to apply the dimensionality reduction procedure twice, in order to get a sum of 2-dimensional integrals that we can evaluate exactly. This brings an extra factor $\frac{1}{n+2}$, giving the following effective Green function

$$\begin{aligned} \tilde{G}_{\text{eff}}(R, \omega) &= \\ &= \frac{1}{4\pi R} \sum_{n=0}^{+\infty} \frac{1}{(n+3)(n+2)n!} \left(\frac{-i\omega R}{c} \right)^n = \\ &= \frac{1}{4\pi R} \left(\left(\frac{1}{(ikR)^2} + \frac{2}{(ikR)^3} \right) e^{-ikR} + \frac{1}{(ikR)^2} - \frac{2}{(ikR)^3} \right), \end{aligned} \quad (9)$$

where we put $k = \frac{\omega}{c}$.

Going back to the time-domain we get

$$G_{\text{eff}}(R, t) = \frac{1}{4\pi R} \left(\left(\left(\frac{c\partial_t^{-1}}{R} \right)^2 + 2 \left(\frac{c\partial_t^{-1}}{R} \right)^3 \right) \delta\left(t - \frac{R}{c}\right) + \left(\left(\frac{c\partial_t^{-1}}{R} \right)^2 - 2 \left(\frac{c\partial_t^{-1}}{R} \right)^3 \right) \delta(t) \right), \quad (10)$$

where we introduced the operator

$$\partial_t^{-1} = \int_{-\infty}^t dt'$$

which is well defined when assuming that all the signals vanish for $t < 0$.

IV. WELL BEHAVIOR OF THE EFFECTIVE GREEN FUNCTION

Putting everything together what we obtain for the time-domain interaction element is

$$I_{mn}(t) = \int_{\Gamma} \int_{\Gamma'} f_m(r) t_n(r') \frac{\delta(t - \frac{R}{c})}{4\pi R} dS dS' * p(t) = \left(\sum_{i,j=1}^3 A_{ij} \int_{\partial\Gamma_i} \int_{\partial\Gamma'_j} G_{\text{eff}}(R, t) dl_i dl'_j + \sum_{\alpha=1}^3 \sum_{\beta=1}^2 \left(B_{\alpha\beta} \int_{\partial\partial\Gamma_{\alpha\beta}} \int_{\Gamma'} G_{\text{eff}}(R, t) dS' + B'_{\alpha\beta} \int_{\Gamma} \int_{\partial\partial\Gamma'_{\alpha\beta}} G_{\text{eff}}(R, t) dS \right) \right) * p(t), \quad (11)$$

where $\partial\Gamma_i$ is the edge i of the corresponding triangle, $\partial\partial\Gamma_{\alpha\beta}$ is the vertex β of the edge α and $A_{ij}, B_{\alpha\beta}$ are matrices of coefficients.

Looking at the explicit form of the effective green function some question may arise when comparing the effective signal, integrated in the 2-dimensional integrals, with the original signal integrated in 4 dimensions.

Without loss of generality we consider a temporal function $p(t)$ which is nonzero only in an interval $(-\delta t, 0)$, and analytic in the same interval. In this case, for fixed time t , the integrand in four dimensions is nonzero only for distances

$$ct < R < c(t + \delta t).$$

Hence, for a sufficiently large value of t the integrand in four dimensions vanishes in the entire domain of integration. The effective signal for large time $t > R$ is not explicitly zero, and this is apparently in contrast with equation (11). However it can be shown that the specific combination of temporal integrals in (10) vanishes after the convolution with $p(t)$. A similar problem arises for small values of t and δt when considering nontouching triangles. For small enough values of $t, \delta t$ the integrand in 4 dimensions is zero, but the effective Green function in this case gives

$$\frac{1}{4\pi R} \left(\left(\frac{c\partial_t^{-1}}{R} \right)^2 + 2 \left(\frac{c\partial_t^{-1}}{R} \right)^3 \right) \delta(t) \quad (12)$$

which is nonzero. This effective signal consists of two terms, which are proportional to $\frac{1}{R^3}$ and $\frac{1}{R^4}$. When considering nontouching triangles, the integration of each one of these terms results to be zero for the specific combination appearing in the right-hand side of (11). The number of terms appearing in (12) is related to the number of times we applied the dimensional reduction procedure, in this case two. This result is independent of the type of temporal function, hence also for small times no contradiction arises.

V. EXACT RESULTS FOR PULSE TEMPORAL FUNCTION

The last step to obtain the exact interaction element is to perform the integration in the right-hand side of (11). Assuming again a temporal function which is analytical in a finite interval and zero otherwise, we can efficiently approximate such function with polynomials. For many of the commonly used temporal discretisation schemes, the temporal basis function is piecewise polynomial and our procedure will yield the exact value of the integral. After the convolution with the effective Green function the resulting integrand becomes of the type

$$G_{\text{eff}}(R, t) * p(t) = \begin{cases} 0 & \text{for } R < ct \\ \sum_{n=-4}^{n_1} a_n(t) R^n & \text{for } ct < R < c(t + \delta t) \\ \sum_{n=-4}^{n_2} b_n(t) R^n & \text{for } R > c(t + \delta t) \end{cases}, \quad (13)$$

where the coefficients $a_n(t), b_n(t)$ are also polynomials in time.

As a first test of the formula we consider a pulse function

$$p(t) = \begin{cases} 0 & \text{for } t < -\delta t \\ 1 & \text{for } -\delta t < t < 0 \\ 0 & \text{for } t > 0 \end{cases}. \quad (14)$$

The convolution with the Green function appearing in the 4-dimensional integral is just

$$G(R, t) * p(t) = \frac{1}{4\pi R} p(t - R) = \begin{cases} 0 & \text{for } R < t \\ \frac{1}{4\pi R} & \text{for } t < R < t + \delta t \\ 0 & \text{for } R > t + \delta t \end{cases} \quad (15)$$

while the convolution with the effective Green function gives

$$G_{\text{eff}}(R, t) * p(t) = \begin{cases} 0 & \text{for } R < t \\ \frac{1}{4\pi} \left(\frac{1}{6R} - \frac{t^2}{2R^3} + \frac{t^3}{3R^4} \right) & \text{for } t < R < t + \delta t \\ \frac{1}{4\pi} \left(\frac{(t + \delta t)^2 - t^2}{2R^3} - \frac{((t + \delta t)^3 - t^3)}{3R^4} \right) & \text{for } R > t + \delta t \end{cases}. \quad (16)$$

where we set $c = 1$ for a more compact formula. In table I we show the results, in units of $c = 1$, for different choices of pairs of triangles and the comparison with a numerical Gaussian integration pushed to 900 points for each triangle. In the first column the type of triangle pair is specified in function of their distance. The small distance pairs are separated by a distance

of the same order of their diameters, while the large distance pairs are separated by a distance of 10^2 times their diameter. In the second and third column the integration of $\frac{1}{R}$ over the entire triangles is reported by means of the exact calculation in the second proposed and with the Gaussian quadrature method respectively. In the fourth and fifth column the exact and the Gaussian calculation of $\frac{1}{R}$ over the intersection with a spherical shell of inner radius t and outer radius $t + \delta t$, that gives the integral of (15) in unit of $c = 1$. First we note that the integrals over full pairs at small distance shows an agreement of the exact and the numerical calculation of the order of $10^{-12}, 10^{-13}$. This is not surprising since the quadrature rule without any singularity works efficiently over the full triangles, but this also shows the precision reachable with the exact method. In the case of the intersection with the shell the relative discrepancy between the two method is around $10^{-3}, 10^{-4}$ for all the data taken. This is imputable to the loss of precision of the quadrature rule. Similarly in presence of singularity we note the same order of discrepancy for the adjacent triangles and an even higher loss of precision for coincident triangles. The large distance integrals over full triangles show a relative discrepancy of 10^{-8} . We impute this lower precision to the exact formula this time for the following reasons. When lowering the number of quadrature points, the quadrature rule returns the exact same result within a margin of 10^{-12} . Instead, the exact formula in this case contains a sum of terms where cancellations between large numbers occurs bringing an estimated absolute error of 10^{-10} that on numbers around 0.02 agree with the relative discrepancy obtained. Such cancellations however are dumped by the smaller δt in case of the intersection with the shell, bringing an estimated absolute error of 10^{-11} . These estimates were given by checking the integral of (12) over the combination (11), which must be zero for nontouching triangles, as we mentioned in the previous section. For the small distance data this integral gives values of the order of 10^{-15} while for the large distance ones gives an order of 10^{-10} when taking δt around the size of the triangles, and an order of 10^{-11} for δt around 0.1 times the size of the triangles. For the coincident and adjacent triangles we cannot check the precision with the Gaussian method but in this case the exact formula can be reduced to two 2 dimensional terms for the coincident triangles and four terms for the adjacent ones. No cancellations between large number occurs in this scenario, and so we expect an accuracy close to machine precision in these cases.

In conclusion these data show the validity of the formula proposed and the extremely high accuracy our closed form solution can deliver. Note that in large scale simulations, compression algorithms will be required, leaving only near field interactions to be computed explicitly. The discussion above demonstrated these interactions can be computed up to extremely high precision.

In terms of efficiency this method requires computation of a higher number of two dimensional integrals for each single double surface integral. However this can be solved by saving all the possible nonzero two dimensional integrals for an entire

Distance	Exact	Gaussian	Ex. with sh.	Gs. with sh.
Coincident	7.7041	7.1042	0.55959	0.55888
Coincident	8.1701	7.5378	0.845036	0.846431
Coincident	9.4508	8.9539	1.3191	1.3167
Adjacent	6.3673	6.3680	2.2098	2.2102
Adjacent	4.1943	4.1972	1.37124	1.37134
Adjacent	6.7795	6.7778	1.6199	1.6189
Small dist.	0.91515418951784	0.91515418951789	0.1140	0.1138
Small dist.	0.7037510311220	0.7037510311218	0.038979	0.038963
Small dist.	0.87033587862231	0.87033587862235	0.3681	0.3679
Large dist.	0.0227688407	0.0227688420	0.0110376	0.0110365
Large dist.	0.024307922	0.024307917	0.0085634	0.0085608
Large dist.	0.0253067110	0.0253067123	0.0110342	0.0110366

TABLE I

mesh, and then combining the values for each double surface integral following the exact formula.

VI. CONCLUSION

In this work a method to obtain a closed form for the interaction matrix elements of time domain boundary integrals equations (TD-BIEs) is proposed. The exact calculation of the matrix elements is obtained by firstly expanding in Taylor series the Green function in the frequency domain. Then a dimensional reduction procedure is applied to the series by using the homogeneity property of the single terms. When going back to the time domain, a new effective time domain Green function is obtained. The resulting integrals with lower dimensionality can be evaluated exactly when the temporal bases function is piecewise polynomial. We showed results for a piecewise constant temporal function, comparing the exact calculation with the numerical Gaussian integration. The results of this method allow to calculate efficiently and precisely the interaction matrix elements and they can be used as a benchmark to evaluate other numerical schemes, as well as to improve stability of the MoT algorithm for TD-BIEs. In this contribution, we limited ourselves to the case of the scalar wave equation. Generalization of this method to the integrands encountered in solving the Maxwell wave equation are subject of ongoing research.

ACKNOWLEDGEMENT

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 101001847)

REFERENCES

- [1] A. G. Polimeridis, F. Vipiana, J. R. Mosig, and D. R. Wilton, "DIRECTFN: Fully Numerical Algorithms for High Precision Computation of Singular Integrals in Galerkin SIE Methods," *IEEE Transactions on Antennas and Propagation*, vol. 61, pp. 3112–3122, June 2013. Conference Name: IEEE Transactions on Antennas and Propagation.
- [2] A. G. Polimeridis, F. Vipiana, J. R. Mosig, and D. R. Wilton, "A general framework for high precision computation of singular integrals in Galerkin SIE formulations," in *2013 IEEE Antennas and Propagation Society International Symposium (APSURSI)*, pp. 454–455, July 2013. ISSN: 1947-1491.
- [3] M. Lenoir and N. Salles, "Evaluation of 3-D Singular and Nearly Singular Integrals in Galerkin BEM for Thin Layers," *SIAM Journal on Scientific Computing*, vol. 34, Jan. 2012.

- [4] D. Tihon and C. Craeye, "All-Analytical Evaluation of the Singular Integrals Involved in the Method of Moments," *IEEE Transactions on Antennas and Propagation*, vol. 66, pp. 1925–1936, Apr. 2018. Conference Name: IEEE Transactions on Antennas and Propagation.
- [5] M.-D. Zhu, X.-L. Zhou, and W.-Y. Yin, "Efficient Evaluation of Double Surface Integrals in Time-Domain Integral Equation Formulations," *IEEE Transactions on Antennas and Propagation*, vol. 61, pp. 4653–4664, Sept. 2013. Conference Name: IEEE Transactions on Antennas and Propagation.
- [6] M.-D. Zhu, Z. Lin, X. Zhao, and Y. Zhang, "Numerical Evaluation of Weakly Near-Singular Integrals in Time-Domain Integral Equations," in *2020 IEEE MTT-S International Conference on Numerical Electromagnetic and Multiphysics Modeling and Optimization (NEMO)*, pp. 1–3, Dec. 2020.
- [7] A. Yucel and A. Ergin, "Exact evaluation of retarded-time potential integrals for the RWG bases," *IEEE Transactions on Antennas and Propagation*, vol. 54, pp. 1496–1502, May 2006. Conference Name: IEEE Transactions on Antennas and Propagation.
- [8] H. A. Ülkü and A. A. Ergin, "Analytical Evaluation of Transient Magnetic Fields Due to RWG Current Bases," *IEEE Transactions on Antennas and Propagation*, vol. 55, pp. 3565–3575, Dec. 2007. Conference Name: IEEE Transactions on Antennas and Propagation.
- [9] Y. Shi, M.-Y. Xia, R.-S. Chen, E. Michielssen, and M. Lu, "Stable Electric Field TDIE Solvers via Quasi-Exact Evaluation of MOT Matrix Elements," *IEEE Transactions on Antennas and Propagation*, vol. 59, pp. 574–585, Feb. 2011. Conference Name: IEEE Transactions on Antennas and Propagation.
- [10] Y. Shi, H. Bağcı, B. Shanker, M. Lu, and E. Michielssen, "Quasi-exact evaluation of time domain MFIE MOT matrix elements," in *2013 IEEE Antennas and Propagation Society International Symposium (APSURSI)*, pp. 41–42, July 2013. ISSN: 1947-1491.
- [11] X. Wang, Y. Shi, M. Lu, B. Shanker, E. Michielssen, and H. Bağcı, "Stable and Accurate Marching-on-in-Time Solvers of Time Domain EFIE, MFIE, and CFIE Based on Quasi-Exact Integration Technique," *IEEE Transactions on Antennas and Propagation*, vol. 69, pp. 2218–2229, Apr. 2021. Conference Name: IEEE Transactions on Antennas and Propagation.
- [12] D. Wilton, S. Rao, A. Glisson, D. Schaubert, O. Al-Bundak, and C. Butler, "Potential integrals for uniform and linear source distributions on polygonal and polyhedral domains," *IEEE Transactions on Antennas and Propagation*, vol. 32, pp. 276–281, Mar. 1984. Conference Name: IEEE Transactions on Antennas and Propagation.